## 14.5: The Chain Rule

Chain Rule for functions of a single variable: If $y = f(x)$ and $x = g(t)$ where $f$ and $g$ are differentiable functions, then $y$ is indirectly a differentiable function of $t$ and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dg}{dt}.$$

$y = f(x) = f(g(t))$

$x = g(t)$

EXAMPLE 1. *Let $z = x^y$, where $x = t^2$, $y = \sin t$. Compute $z'(t)$.*

$$z = x^y = (t^2)^{\sin t} = \boxed{t^{2\sin t} = z(t)}$$

In Calc 1 we did logarithmic differentiation:

$$\Leftarrow \left(\ln\left(t^{2\sin t}\right)\right)' = (\ln z(t))'$$

$$(2\sin t \ln t)' = \frac{z'}{z}$$

and so on - -

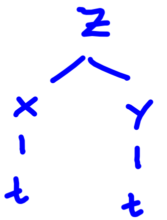
Assume that all functions below have continuous derivatives (ordinary or partial).

- CASE 1: $z = f(x, y)$, where $x = x(t)$, $y = y(t)$ and compute $z'(t)$.

  $z = f(x,y) = f(x(t), y(t))$

  *Chain Rule:*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$



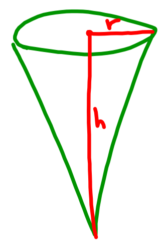

SOLUTION OF EXAMPLE 1:

$z = x^y$, $x = t^2$, $y = \sin t$

$$\frac{dz}{dt} = z_x \cdot x' + z_y \cdot y' = y x^{y-1} \cdot 2t + x^y \ln x \cdot \cos t$$

$$= 2t(\sin t)\, t^{2(\sin t - 1)} + t^{2\sin t}(\ln t^2)\cos t$$

EXAMPLE 2. *The radius of a right circular cone is increasing at a rate of 1.8 cm/s while its height is decreasing at a rate 2.5 cm/s. At what rate is the volume of the cone changing when the radius is 120 cm and the height is 140 cm.*




Given $\frac{dr}{dt} = 1.8$ cm/s

$\frac{dh}{dt} = -2.5$ cm/s

Find $\left.\frac{dV}{dt}\right|_{\substack{r=120\text{ cm}\\ h=140\text{ cm}}} = ?$

$$V = \frac{1}{3}\pi r^2 h$$

$$V(t) = \frac{\pi}{3}[r(t)]^2 \cdot h(t)$$

By Chain Rule:

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt} =$$

$$= \frac{2\pi}{3} r h \cdot \frac{dr}{dt} + \frac{\pi}{3} r^2 \frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{\substack{r=120\\ h=140}} = \frac{2\pi}{3}\cdot 120\cdot 140\cdot \underline{1.8} + \frac{\pi}{3}\cdot 120^2(\underline{-2.5})$$

$$= \frac{\pi}{3}\left[2\cdot 120\cdot \cancel{140}^{28}\cdot \frac{9}{\cancel{5}_1} - 120\cdot \cancel{120}^{60}\cdot \frac{5}{\cancel{2}}\right]$$

$$= \frac{\pi}{\cancel{3}_1}\cdot \cancel{120}^{40}\left[2\cdot 28\cdot 9 - 60\cdot 5\right]$$

$$= 40\pi\cdot 6\left[28\cdot 3 - 50\right] = 240\cdot 34\pi = \boxed{?}\ \text{cm}^3/\text{s}$$

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$z = f(x,y) = f(x(s,t), y(s, t))$

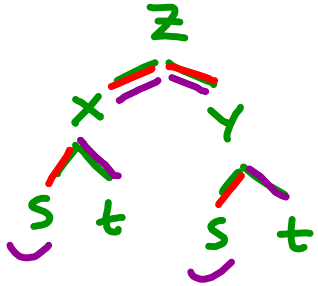
- CASE 2: $z = f(x,y)$, where $x = x(s,t)$, $y = y(s,t)$ and compute $z_s$ and $z_t$.

  *Chain Rule:*

  $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

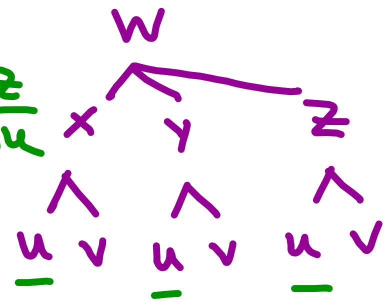
  $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

  *Tree diagram:*



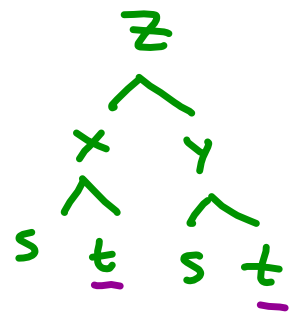

EXAMPLE 3. *Write out the Chain Rule for the case where* $w = f(x,y,z)$ *and* $x = x(u,v)$, $y = y(u,v)$ *and* $z = z(u,v)$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$




$$\frac{\partial w}{\partial v} =$$

EXAMPLE 4. *If* $z = \sin x \cos y$, *where* $x = (s-t)^2$, $y = s^2 - t^2$ *find* $z_s + z_t$.




$$z_s = z_x \cdot x_s + z_y \cdot y_s$$

$$= \cos x \cos y \; 2(s-t) + \sin x(-\sin y) \cdot 2s$$

$$z_t = z_x \cdot x_t + z_y \cdot y_t$$

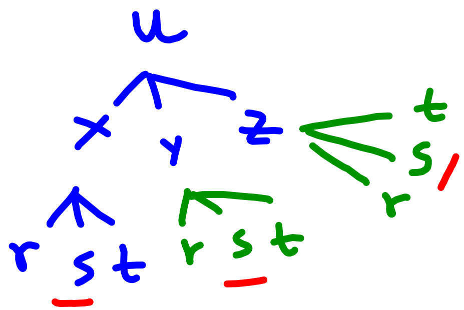
$$= \cos x \cos y \cdot (-2(s-t)) + \sin x(-\sin y) \cdot (-2t)$$

$$z_s + z_t = \sin x(-\sin y) \cdot 2s + \sin x(-\sin y) \cdot (-2t)$$

$$= -2 \sin x \sin y (s-t)$$

$$= -2 \sin(s-t)^2 \cdot (\sin(s^2 - t^2))(s-t)$$

EXAMPLE 5. *If* $u = x^2y + y^3z^2$ *where* $x = rse^t$, $y = r + s^2e^{-t}$, $z = rs\sin t$, *find* $u_s$ *when* $(r, s, t) = (1, 2, 0)$




$$u_s = u_x \cdot x_s + u_y \cdot y_s + u_z \cdot z_s$$

$$= 2xy\, re^t + (x^2 + 3y^2z^2) \cdot 2se^{-t} +$$

$$+ 2y^3z \cdot r\sin t$$

$$u_s(1,2,0) = 2 \cdot 2 \cdot 5 \cdot 1 \cdot e^0 + (2^2 + 0) \cdot 2 \cdot 2e^{-0} + 0$$

$$= 20 + 16 = \boxed{36}$$

$x(1,2,0) = 1 \cdot 2 \cdot e^0 = 2$

$y(1,2,0) = 1 + 2^2e^{-0} = 5$

$z(1,2,0) = 0$

**Implicit differentiation:** Suppose that an equation

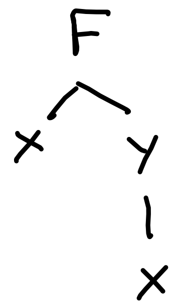
$$F(x, y) = 0^{c}$$

defines $y$ implicitly as a differentiable function of $x$, i.e. $y = y(x)$, where $F(x, y(x)) = 0$ for all $x$ in the domain of $y(x)$. Find $y'$:

$$F(x, y) = 0^{c}$$

$$F(x, y(x)) = 0$$

$$\frac{dF(x, y(x))}{dx} = \frac{d0}{dx}$$




$$F_x + F_y \cdot y'(x) = 0 \Rightarrow y'(x) = \boxed{-\frac{F_x}{F_y} = \frac{dy}{dx}}$$

EXAMPLE 6. *Find* $y'$ *if* $x^4 + y^3 = 6e^{xy}$.

defined implicitely

$$\underbrace{x^4 + y^3 - 6e^{xy}}_{F(x,y)} = 0$$

$$y' = \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{4x^3 - 6ye^{xy}}{3y^2 - 6xe^{xy}}$$

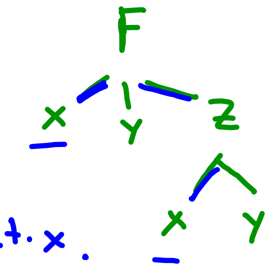
Suppose that an equation

$$F(x, y, z) = 0$$

defines $z$ implicitly as a differentiable function of $x$ and $y$, i.e. $z = z(x, y)$, where

$$F(x, y, z(x, y)) = 0$$

for all $(x, y)$ in the domain of $z$. Find the partial derivatives $z_x$ and $z_y$:




$$F(x, y, z(x, y)) = 0$$

To find $z_x$ differentiate both sides w.r.t. $x$.

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \text{or} \quad \boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$

Similarly one can get

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

EXAMPLE 7. *If* $x^4 + y^3 + z^2 + xye^z = 10$ *find*

$$F(x,y,z) = x^4 + y^3 + z^2 + xye^z - 10$$

**(a)** $z_x$ *and* $z_y$

$$z_x = -\frac{F_x}{F_z} = -\frac{4x^3 + ye^z}{2z + xye^z}$$

$$z_y = -\frac{F_y}{F_z} = -\frac{3y^2 + xe^z}{2z + xye^z}$$

**(b)** $x_y$ *and* $x_z$

$$x_y = \frac{\partial x}{\partial y} = -\frac{F_y}{F_x} = \ldots$$

$$x_z = \frac{\partial x}{\partial z} = -\frac{F_z}{F_x} = \ldots$$