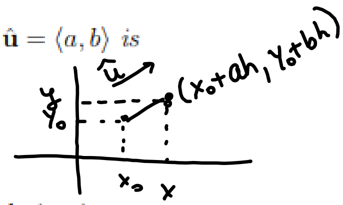
## 14.6: Directional Derivatives and the Gradient Vector

Recall that the two partial derivatives $f_x(x,y)$ and $f_y(x,y)$ of $f(x,y)$ represent the rate of change of $f$ as we vary $x$ (holding $y$ fixed) and as we vary $y$ (holding $x$ fixed) respectively. In other words, $f_x(x,y)$ and $f_y(x,y)$ represent the rate of change of $f$ in the directions of the unit vectors **i** and **j** respectively. Let's consider how to find the rate of change of $f$ if we allow both $x$ and $y$ to change simultaneously, or how to find the rate of change of $f$ in the direction of an arbitrary vector **u**.

DEFINITION 1. *The rate of change of $f(x,y)$ in the direction of the unit vector $\hat{\mathbf{u}} = \langle a,b\rangle$ is called the* **directional derivative** *and it is denoted by $D_{\mathbf{u}}f(x,y)$.*

*The* **directional derivative** *of $f$ at $(x_0,y_0)$ in the direction of the unit vector $\hat{\mathbf{u}} = \langle a,b\rangle$ is*

$$D_{\mathbf{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$




*if this limit exists.*

REMARK 2. By comparing the last definition with the definitions of the partial derivatives:

$$f_x(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}, \quad f_y(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0,y_0+h) - f(x_0,y_0)}{h}$$

we see that $\hat{u} = \hat{i}$ $\qquad$ $\hat{u} = \hat{j} = (0,1)$

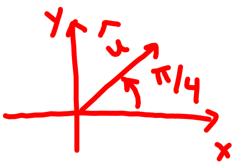
$f_x(x_0,y_0) = D_{\hat{i}}f(x_0,y_0)$ and $f_y(x_0,y_0) = D_{\hat{j}}f(x_0,y_0)$

For computational purposes use the following theorem.

THEOREM 3. *If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $\hat{\mathbf{u}} = \langle a, b\rangle$ and*

$$\boxed{D_{\mathbf{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b.} = \overbrace{\langle f_x, f_y\rangle}^{\nabla f} \cdot \underbrace{\langle a, b\rangle}_{\hat{u}}$$

EXAMPLE 4. *Find the rate of change $f(x,y) = x^3 + \sin(xy)$ at the point $(1, \pi/2)$ in the direction indicated by the angle $\theta = \pi/4$.*




$\vec{u} = \langle 1, 1\rangle$

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \langle \underbrace{\tfrac{\sqrt{2}}{2}}_{=a}, \underbrace{\tfrac{\sqrt{2}}{2}}_{=b} \rangle = \langle \cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{4}\rangle$$

$f_x = 3x^2 + y\cos(xy)$

$f_y = x\cos(xy)$

$f_x(1, \frac{\pi}{2}) = 3$

$f_y(1, \frac{\pi}{2}) = 0$

We should find here a directional derivative at $(1, \frac{\pi}{2})$ in the direction of $\hat{u}$.

By Th.3 we get

$$D_{\hat{u}} = f_x(1, \tfrac{\pi}{2}) \cdot \frac{\sqrt{2}}{2} + f_y(1, \tfrac{\pi}{2}) \frac{\sqrt{2}}{2}$$

$$= 3\frac{\sqrt{2}}{2} + 0 \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{3\sqrt{2}}{2}}$$

***The Directional Derivative As The Dot Product Of Two Vectors. Gradient.***

DEFINITION 5. *The* **gradient** *of* $f(x,y)$ *is the vector function* $\nabla f$ *defined by*

$$\text{grad } f(x,y) = \nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Notations for gradient: **grad**$f$ or $\nabla f$ which is read "del $f$".

EXAMPLE 6. *Find the gradient of* $f = \cos(xy) + e^x$ *at (0,3).*

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \langle -y\sin(xy) + e^x, -x\sin(xy) \rangle$$

$$\nabla f(0,3) = \langle 1, 0 \rangle$$

By Theorem 3 we have:

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a, b\rangle$$
$$= \nabla f \cdot \hat{u}$$

Formula for the directional derivative using the gradient vector:

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \hat{\mathbf{u}}.$$

or $D_u f = \nabla f \cdot \hat{u}$

EXAMPLE 7. *Find the directional derivative for $f$ from Example 6 at* $(0,3)$ *in the direction* $\langle 3,4\rangle$.

$f(x,y) = \cos(xy) + e^x$

$\nabla f(0,3) = \langle 1,0\rangle$ by Ex. 3.

$\vec{u} = \langle 3,4\rangle$

Normalize $\vec{u}$ : $|\vec{u}| = \sqrt{3^2+4^2} = 5$

$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \boxed{\frac{\langle 3,4\rangle}{5}} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$

$$D_{\vec{u}} f(0,3) = \nabla f(0,3)\cdot \hat{u} = \langle 1,0\rangle \cdot \frac{\langle 3,4\rangle}{5}$$

$$= \frac{1\cdot 3 + 0\cdot 4}{5} = \boxed{\frac{3}{5}}$$

**The directional derivative of function of *three* variables**

THEOREM 8. *If $f$ is a differentiable function of $x$, $y$ and $z$, then $f$ has a directional derivative in the direction of any unit vector $\hat{\mathbf{u}} = \langle a, b, c\rangle$ and*

$$D_{\mathbf{u}}f(x,y) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c = \boxed{\nabla f \cdot \hat{\mathbf{u}},}$$

*where*

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

*is the gradient vector of $f(x,y,z)$.*

EXAMPLE 9. *Find the directional derivative of* $f(x,y,z) = z^3 - x^2y$ *at the point* $(1,6,2)$ *in the direction* $\mathbf{u} = \langle 1, -2, 3\rangle$.

Find gradient at $(1,6,2)$:

$$\left.\begin{array}{lll} f_x = -2xy & \Rightarrow & f_x(1,6,2) = -12 \\ f_y = -x^2 & \Rightarrow & f_y(1,6,2) = -1 \\ f_z = 3z^2 & \Rightarrow & f_z = 12 \end{array}\right\} \Rightarrow \nabla f(1,6,2) = \langle -12, -1, 12\rangle$$

Normalize $\vec{u} = \langle 1,-2,3\rangle$:

$$|\vec{u}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$$

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\langle 1,-2,3\rangle}{\sqrt{14}}$$

Apply the formula:

$$D_{\vec{u}} f(1,6,2) = \nabla f(1,6,2) \cdot \hat{u} = \langle -12,-1,12\rangle \cdot \frac{\langle 1,-2,3\rangle}{\sqrt{14}}$$

$$= \frac{-12\cdot 1 + (-1)\cdot(-2) + 12\cdot 3}{\sqrt{14}}$$

$$= \frac{26}{\sqrt{14}}\frac{\sqrt{14}}{\sqrt{14}} = \frac{26\sqrt{14}}{14} = \boxed{\frac{13\sqrt{14}}{7}}$$

*QUESTION:* In which of all possible directions does $f$ change fastest and what is the maximum rate of change.

*ANSWER* is provided by the following theorem:

THEOREM 10. *Suppose $f$ is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f$ is $\underbrace{|\nabla f|}$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f$.*

*Proof.*

$$D_{\vec{u}} f = \nabla f \cdot \hat{u} = |\nabla f| \cdot \underbrace{|\hat{u}|}_{=1} \underbrace{\cos\alpha}$$

where $\alpha = \measuredangle(\nabla f, \vec{u})$

$$\max_{\vec{u}} D_{\vec{u}} f = |\nabla f| \cdot \underbrace{\max_{\alpha} \cos\alpha}_{\text{attained when } \alpha=0 \text{ and equal to } 1.} = \boxed{|\nabla f|.}$$

$\alpha = 0$ means that $\nabla f$ and $\vec{u}$ are in the same direction. $\square$

BTW $\min_{\vec{u}} D_{\vec{u}} \vec{f} = -|\nabla f|$ (when $\alpha = \pi$, i.e. $\nabla f$ and $\vec{u}$ have opposite direction).

EXAMPLE 11. *Suppose that the temperature at a point* $(x, y, z)$ *in the space is given by*

$$T(x, y, z) = \frac{100}{1 + x^2 + y^2 + z^2},$$

*where* $T$ *is measured in* $^\circ C$ *and* $x, y, z$ *in meters.*

**(a)** *In which direction does the temperature increase fastest at the point* $(1, 1, -1)$*?*

in the direction of $\nabla T(1,1,-1)$.

$$T_x = \frac{-100 \cdot 2x}{(1+x^2+y^2+z^2)^2} \Rightarrow T_x(1,1,-1) = -\frac{200}{16} = -\frac{25}{2}$$

$$T_y = \frac{-200y}{(1+x^2+y^2+z^2)^2} \Rightarrow T_y(1,1,-1) = -\frac{25}{2}$$

$$T_z = \frac{-200z}{(1+x^2+y^2+z^2)^2} \Rightarrow T_z(1,1,-1) = \frac{25}{2}$$

$$\nabla T(1,1,-1) = \boxed{\left\langle -\frac{25}{2}, -\frac{25}{2}, \frac{25}{2} \right\rangle} = \frac{25}{2}\langle -1,-1,1\rangle$$

**(b)** *What is the maximum rate of increase?* at $(1,1,-1)$?

$$|\nabla f(1,1,-1)| = \left|\frac{25}{2}\langle -1,-1,1\rangle\right| = \frac{25}{2}|\langle -1,-1,1\rangle| = \boxed{\frac{25\sqrt{3}}{2}}$$

$\parallel$
maximal directional derivative at $(1,1,-1)$

**Tangent planes to level surfaces:**

*FACT:* The gradient vector $\nabla F(x_0, y_0, z_0)$ is **orthogonal** to the level surface $\boxed{F(x, y, z) = k}$ at the point $(x_0, y_0, z_0)$.

So, the *tangent plane* to the surface $f(x, y, z) = k$ at the point $(x_0, y_0, z_0)$ has the equation:

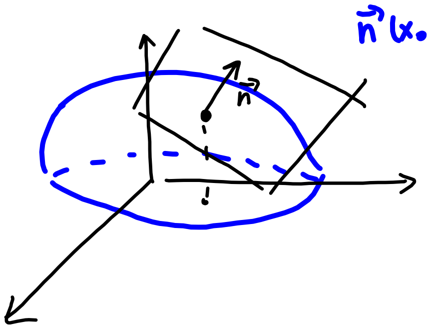
$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

normal to the tangent plane $\vec{n} \parallel \nabla F(x_0, y_0, z_0)$

The normal line to the surface at the point $(x_0, y_0, z_0)$ is the line passing through $(x_0, y_0, z_0)$ and perpendicular to the tangent plane. Therefore its direction is given by the $\vec{v} \parallel \vec{n}(x_0, y_0, z_0)$ vector

$\vec{n}(x_0, y_0, z_0) \parallel \nabla F(x_0, y_0, z_0)$ or $\vec{v} \parallel \nabla F(x_0, y_0, z_0)$.



EXAMPLE 12. *Find the equation of the tangent plane and normal line at the point* $(\underset{=x_0}{1}, \underset{=y_0}{0}, \underset{=z_0}{5})$ *to the surface* $xe^{yz} = 1$.

$F(x,y,z) = xe^{yz}$

Find $\nabla F(1,0,5)$:

$F_x = e^{yz} \Rightarrow F_x(1,0,5) = 1$

$F_y = xze^{yz} \Rightarrow F_y(1,0,5) = 5$

$F_z = xye^{yz} \Rightarrow F_z(1,0,5) = 0$

$\Rightarrow \nabla F(1,0,5) = \langle 1,5,0 \rangle$

So, normal to the tangent plane at $(1,0,5)$ is

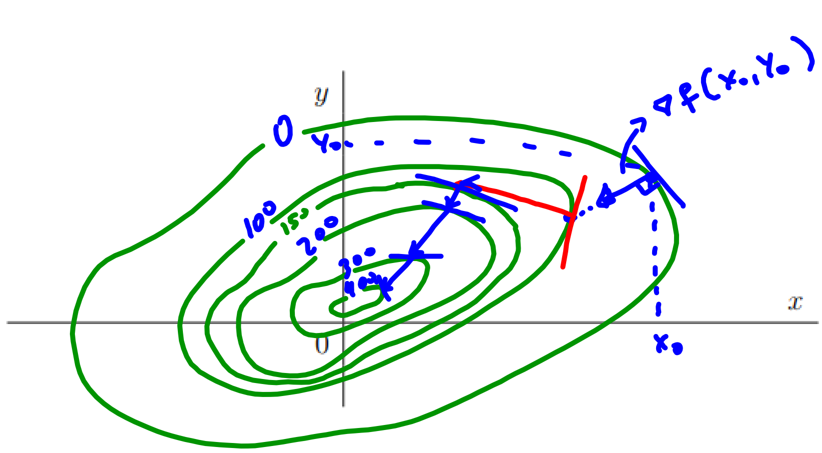
$\vec{n}(1,0,5) = \langle \underset{=a}{1}, \underset{=b}{5}, \underset{=c}{0} \rangle$.

$1 \cdot (x-1) + 5(y-0) + 0 \cdot (z-5) = 0$

$x - 1 + 5y = 0$

$\boxed{x + 5y = 1}$

Likewise, the gradient vector $\nabla f(x_0, y_0)$ is **orthogonal** to the level curve $f(x, y) = k$ at the point $(x_0, y_0)$.




Consider a topographical map of a hill and let $f(x, y)$ represents the height above sea level at a point with coordinates $(x, y)$. Draw a curve of steepest ascent.