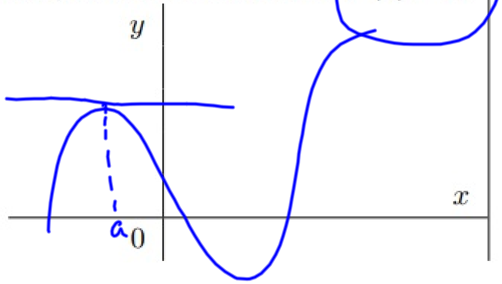
12.7: Maximum and minimum values

Function $y = f(x)$

DEFINITION 1. *A function $f(x)$ has a local maximum at $x = a$ if $f(a) \geq f(x)$ when x is near a (i.e. in a neighborhood of a). A function f has a local minimum at $x = a$ if $f(a) \leq f(x)$ when x is near a.*

If the inequalities in this definition hold for ALL points x in the domain of f, then f has an **absolute max** (or **absolute min**) at a

If the graph of f has a tangent line at a local extremum, then the tangent line is horizontal: $f'(a) = 0$.



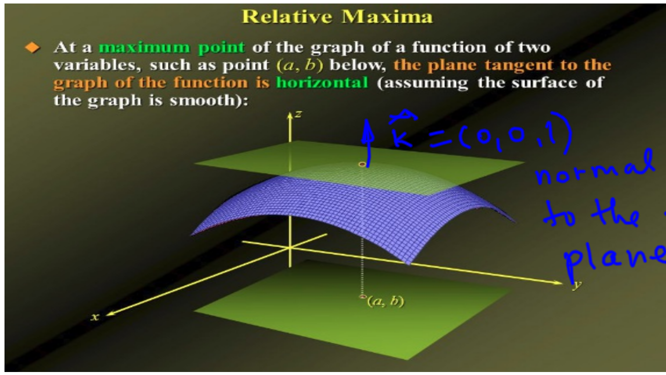
Function of two variables $z = f(x, y)$

relative

DEFINITION 2. *A function $f(x, y)$ has a local maximum at $(x, y) = (a, b)$ if $f(a, b) \geq f(x, y)$ when (x, y) is near (a, b) (i.e. in a neighborhood of (a, b)). A function f has a local minimum at $(x, y) = (a, b)$ if $f(a, b) \leq f(x, y)$ when (x, y) is near (a, b).*

If the inequalities in this definition hold for ALL points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

If the graph of f has a tangent plane at a local extremum, then the tangent PLANE is horizontal.



normal to the tangent plane

THEOREM 3. *If f has a local extremum (that is, a local maximum or minimum) at (a, b) and the first-order partial derivatives exist there, then*

$$f_x(a,b) = f_y(a,b) = 0 \quad (or,\ equivalently, \nabla f(a,b) = 0.)$$

Proof Tangent plane to the graph of the function $z = f(x,y)$ has a normal vector

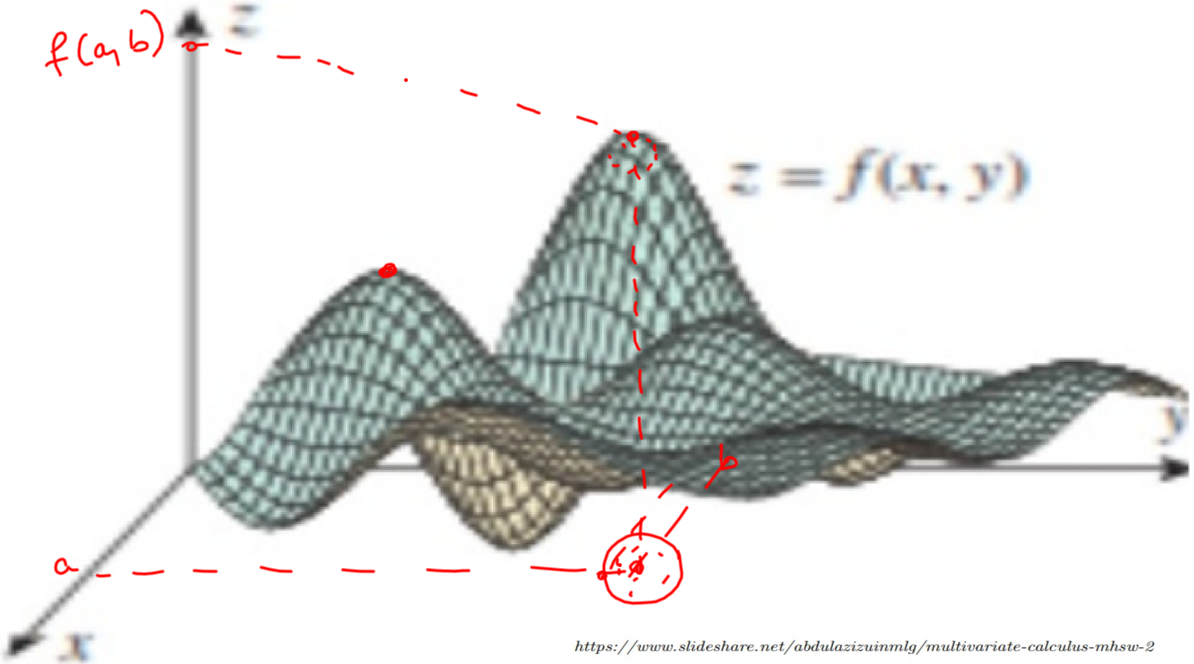
$$\vec{n} = \langle f_x, f_y, -1 \rangle.$$

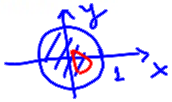
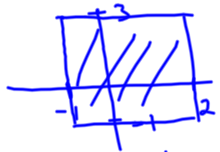
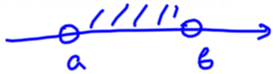
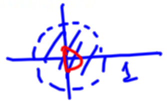
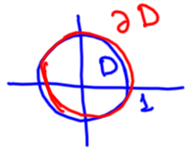
At the point (a,b) we have $\vec{n}(a,b) = \langle f_x(a,b), f_y(a,b), -1 \rangle$ and this vector must be parallel to $\hat{k} = \langle 0,0,1 \rangle$ (because tangent plane at (a,b) is horizontal). So, $f_x(a,b) = f_y(a,b) = 0$. $\square$

DEFINITION 4. *A point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or one of this partial derivatives does not exist, is called a* **critical point** *of f.*

At a critical point, a function could have a local max or a local min, or neither. We will be concerned with two important questions:

- Are there any local or absolute extrema?
- If so, where are they located?

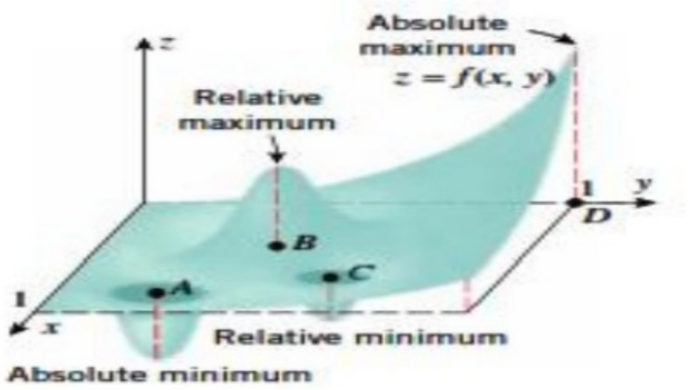


in $\mathbb{R}$	in $\mathbb{R}^2$
closed interval $[a, b]$  $\{x \in \mathbb{R} \mid a \le x \le b\}$	closed set   $D=\{(x,y) \mid x^2+y^2 \le 1\}$ $\{(x,y) \mid -1 \le x \le 2, -1 \le y \le 3\}$
open interval (a, b)  $\{x \in \mathbb{R} \mid a < x < b\}$	open set   $D=\{(x,y) \mid x^2+y^2 < 1\}$ $\{(x,y) \mid -1 < x < 2, -1 < y < 3\}$
end points of an interval $x = a$, $x = b$	boundary points (usually a curve)  $\partial D = \{(x,y) \mid x^2+y^2 = 1\}$

DEFINITION 5. *A* **bounded set** *in* $\mathbb{R}^2$ *is one that contained in some disk.*

THE EXTREME VALUE THEOREM:

Function $y = f(x)$	Function of two variables $z = f(x, y)$
If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(x_1)$ and an absolute minimum value $f(x_2)$ at some points x_1 and x_2 in $[a, b]$.	If f is continuous on a closed bounded set $\mathcal{D}$ in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in $\mathcal{D}$.



https://www.slideshare.net/abdulazizuinmlg/multivariate-calculus-mhsw-2

EXAMPLE 6. *Find extreme values of* $f(x, y) = x^2 + y^2$.

	Local	Absolute
Maximum	none	none
Minimum	at $(0,0)$	at $(0,0)$

Domain: $\mathbb{R}^2$

$f_x = 2x = 0$

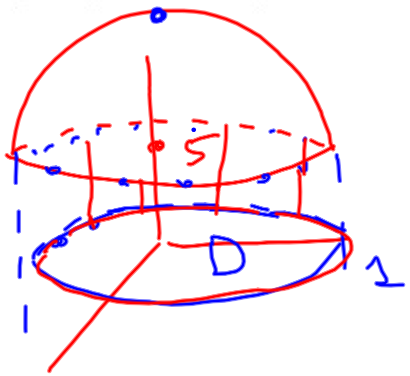
$f_y = 2y = 0$ $\Rightarrow (0,0)$ is a critical point

EXAMPLE 7. *Find extreme values of* $f(x, y) = 5 + \sqrt{1 - x^2 - y^2}$.

	Local	Absolute
Maximum	at $(0,0)$	at $(0,0)$
Minimum	none	on the ∂D

Domain: $\{(x,y) \mid x^2 + y^2 \leq 1\}$

closed and bdd



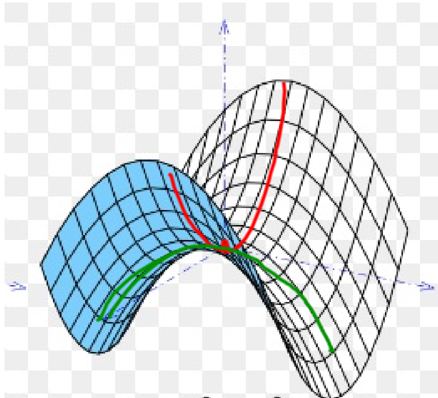
EXAMPLE 8. *Find extreme values of* $f(x,y) = x^2 - y^2$.

	Local	Absolute
Maximum	no	no
Minimum	no	no

Domain: $\mathbb{R}^2$

$f_x = 2x$
$f_y = -2y$ $\Big\} \Rightarrow (0,0)$ crit. point

$f(x,0) = x^2$
$f(0,y) = -y^2$



$z = x^2 - y^2$

REMARK 9. Example 8 illustrates so called **saddle point** of f. Note that the graph of f crosses its tangent plane at (a,b).

no local max
and no local min

ABSOLUTE MAXIMUM AND MINIMUM VALUES on a closed bounded set.

THE EXTREME VALUE THEOREM:

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical points of f in (a, b).

2. Find the values of f at the endpoints of the interval.

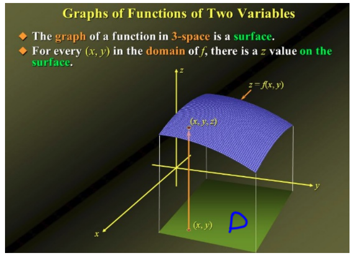
3. The largest of the values from steps 1&2 is the absolute max value; the smallest of the values from steps 1&2 is the absolute min value.

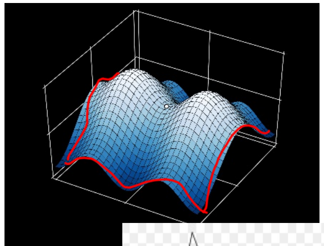
To find the absolute max and min values of a continuous function f on a closed bounded set D:

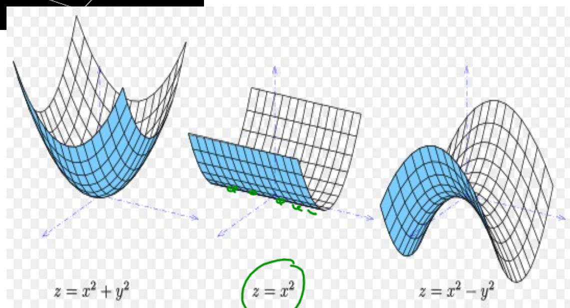
1. Find the values of f at the critical points of f in D.

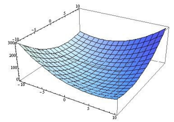
2. Find the extreme values of f on the boundary of D. (This usually involves the Calculus I approach for this work.)

3. The largest of the values from steps 1&2 is the absolute maximum value; the smallest of the values from steps 1&2 is the absolute minimum value.









$$\left.\begin{aligned} z_x &= 2x \\ z_y &= 0 \end{aligned}\right\} \Rightarrow = 0$$

$(0, y)$
critical points

- The quantity to me maximized/minimized is expressed in terms of variables (as few as possible!)
- Any constraints that are presented in the problem are used to reduce the number of variables to the point they are independent,
- After computing partial derivatives and setting them equal to zero you get purely algebraic problem (but it may be hard.)
- Sort out extreme values to answer the original question.

Parameterize curves in $\mathbb{R}^2$:

line through $(1,2)$ in the direction $(4,5)$:

$$\boxed{x = 1+4t \ , \ y = 2+5t}$$

$$\frac{x-1}{4} = \frac{y-2}{5} \Rightarrow \frac{y}{5} - \frac{2}{5} = \frac{x-1}{4}$$

$$\frac{y}{5} = \frac{x-1}{4} + \frac{2}{5}$$

$$x = t \ , \ y = \frac{5}{4}(t-1) + 2$$

Parabola $y = x^2$

$$x = t \ , \quad y = t^2$$

Other curve given explicitely $y = x^2$, $a \le x \le b$

$$x = t, \quad y = t^2, \quad \underbrace{a \le t \le b}_{\text{parameter domain.}}$$

Circle with radius 1: $x^2 + y^2 = 1$

$$x = \cos t \ , \ y = \sin t \ , \quad 0 \le t \le 2\pi \ .$$

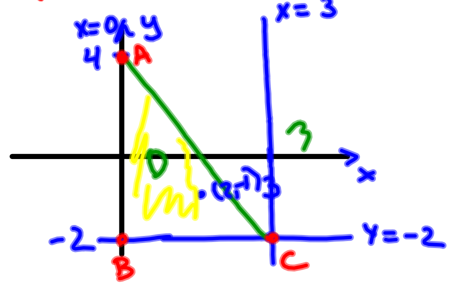
EXAMPLE 10. *A lamina occupies the region* $D = \{(x,y):\ 0 \le x \le 3,\ \ -2 \le y \le 4-2x\}$. *The temperature at each point of the lamina is given by*

$$T(x,y) = 4(x^2 + xy + 2y^2 - 3x + 2y) + 10.$$

Find the hottest and coldest points of the lamina.

In other words, find the absolute maximum and absolute minimum of the function $z = T(x,y)$ on the region D.

STEP 1 Draw the domain:



STEP 2 Find critical points of $T(x,y)$ in D.

$$T_x = 4(2x + y - 3) = 0$$

$$T_y = 4(x + 4y + 2) = 0$$

Solve the system

$$\begin{cases} 2x + y - 3 = 0 \\ x + 4y + 2 = 0 \quad \times(-2) \end{cases}$$

or

$$\begin{cases} 2x + y - 3 = 0 \\ + \\ -2x - 8y - 4 = 0 \end{cases}$$

$$-7y - 7 = 0 \Rightarrow y = -1$$

$$x = -2 - 4y = -2 - 4(-1) = -2 + 4 = 2$$

So, $(2,-1)$ is a critical point of T

$(2,-1) \in D$. So calculat T at $(2,-1)$:

$$T(2,-1) = 4(4 - 2 + 2 - 6 - 2) + 10$$

$$T(2,-1) = \boxed{-6}$$

STEP 3 Describe the boundary

$$\partial D \doteq \overline{AB} \cup \overline{BC} \cup \overline{CA}$$

STEP 4 Calculate T at the endpoints (vertices) of the boundaries:

$$T(A) = T(0,4) = 4\cdot(32+8)+10 = \boxed{170}$$

$$T(B) = T(0,-2) = 4(8-4)+10 = \boxed{26}$$

$$T(C) = T(3,-2) =$$

$$= 4(\cancel{9} - 6 + 8 - \cancel{9} - 4) + 10 = \boxed{2}$$

STEP 5 Find critical points on the ∂D (boundary of D):

$\overline{AB}$	$\overline{BC}$	$\overline{CA}$

Parameterize + parameter domain

$x=0,\ y=t$ $-2<t<4$	$x=t,\ y=-2$ $0<t<3$	$x=t,\ y=4-2t$ $0<t<3$

Find $T|_{\partial D}$

$T(0,t)=4(2t^2+2t)+10$	$T(t,-2)=$ $=4(t^2-2t+8-3t-4)+10$ $=4(t^2-5t+4)+10$	$T(t,4-2t)=$ $4(t^2+t(4-2t)+$ $2(4-2t)^2-3t+$ $+2(4-2t))+10$ $=4(7t^2-35t+40)+10$

Find critical points solving $\frac{\partial T|_{\partial D}}{\partial t}$

$4(4t+2)=0$ $t=-\frac{1}{2}\in(-2,4)$	$4(2t-5)=0$ $t=\frac{5}{2}\in(0,3)$	$4(14t-35)=0$ $14t=35$ $t=\frac{35}{14}=\frac{5}{2}\in(0,3)$

Find T at these critical points:

$$T\left(0,-\tfrac{1}{2}\right)=4\left(2\cdot\tfrac{1}{4}+2\cdot\left(-\tfrac{1}{2}\right)\right)+10$$
$$=4\left(\tfrac{1}{2}-1\right)+10=-2+10=\boxed{8}$$

$$T\left(\tfrac{5}{2},-2\right)=4\left(\left(\tfrac{5}{2}\right)^2-5\cdot\tfrac{5}{2}+4\right)+10$$
$$=4\left(\tfrac{25}{4}-\tfrac{25}{2}+4\right)+10$$
$$=4\left(-\tfrac{25}{4}+4\right)+10$$
$$=4\left(\tfrac{-25+16}{4}\right)+10=\boxed{1}$$

$T\left(\frac{5}{2},\ 4-2\cdot\frac{5}{2}\right) =$

$= \cancel{4}\left(\frac{7\cdot 25}{\cancel{4}} - \frac{35\cdot 5\cdot 2}{\cancel{2\cdot 2}} + \frac{40\cdot 4}{\cancel{4}}\right) + 10$

$= 35\cdot 5 - 35\cdot 10 + 160 + 10$

$= -35\cdot 5 + 170 = -175 + 170 = \boxed{-5}$

$(2,-1)$	A	B	C	$(0,-\frac{1}{2})$	$(\frac{5}{2},-2)$	$(\frac{5}{2},-1)$
-6	170	26	2	8	1	-5

the hottest point is $(0,4)$ with $T = 170°$

the coldest point is $(2,-1)$ with $T = -6$.

Local/Relative Extrema

Second derivatives test:

Suppose f'' is continuous near a and $f'(c) = 0$ (i.e. a is a critical point).	Suppose that the second partial derivatives of f are continuous near (a, b) and $\nabla f(a, b) = \mathbf{0}$ (i.e. (a, b) is a critical point). Let $\mathcal{D} = \mathcal{D}(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$
• If $f''(c) > 0$ then $f(c)$ is a local minimum.	• If $\mathcal{D} > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
• If $f''(c) < 0$ then $f(c)$ is a local maximum.	• If $\mathcal{D} > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
	• If $\mathcal{D} < 0$ then $f(a, b)$ is not a local extremum (saddle point).

NOTE:

• If $f''(c) = 0$, then the test gives no information.	If $\mathcal{D} = 0$ or does not exist, then the test gives no information. fails.

To remember formula for $\mathcal{D}$:

$$\mathcal{D} = f_{xx} f_{yy} - [f_{xy}]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

EXAMPLE 11. *Use the Second Derivative Test to confirm that a local cold point of the lamina in the previous Example is* $(2, -1)$.

$$T_x = 4(2x + y - 3)$$

$$T_y = 4(x + 4y + 2)$$

$T_{xx} = 4 \cdot 2 = 8 > 0$ $\quad T_{xy} = 4$

$T_{yy} = 16$

$$\mathcal{D} = \begin{vmatrix} T_{xx} & T_{xy} \\ T_{xy} & T_{yy} \end{vmatrix} = \begin{vmatrix} 8 & 4 \\ 4 & 16 \end{vmatrix} = 8 \cdot 16 - 16 > 0$$

So, at $(2,-1)$ there is a local minimum, i.e. $(2,-1)$ is the point of a local cold of the lamina.

EXAMPLE 12. *The surface $z = 4xy - x^4 - y^4$ has two peaks and one pass. Find them.*

In other words find two local maximums and a saddle point of this function.

Step 1 Find critical points:

$$z_x = 4y - 4x^3 = 0$$

$$z_y = 4x - 4y^3 = 0$$

or $\left.\begin{array}{l} y = x^3 \\ x = y^3 \end{array}\right\}$

$$y = (y^3)^3$$

$$y - y^9 = 0$$

$$y(1 - y^8) = 0$$

$$y(1-y^4)(1+y^4) = 0$$

$$y(1-y^2)(1+y^2)(1+y^4) = 0$$

$$y(1-y)(1+y)\underbrace{(1+y^2)(1+y^4)}_{\text{no solutions}} = 0$$

$y=0$ $\Downarrow$ $x=0$ $y=1$ $\Downarrow$ $x=1$ $y=-1$ $\Downarrow$ $x=-1$

STEP 2

Find second order derivatives:

$$z_{xx} = -12x^2$$

$$z_{xy} = 4$$

$$z_{yy} = -12y^2$$

STEP 3 Apply the test:

	(0,0)	(1,1)	(-1,-1)
$z_{xx} = -12x^2$	0	$-12 < 0$	$-12 < 0$
$z_{xy} = 4$	4	4	4
$z_{yy} = -12y^2$	0	-12	-12
$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{vmatrix}$	$\begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0$ saddle pass	$\begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 > 0$ local max peak	$\begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} > 0$ local max peak