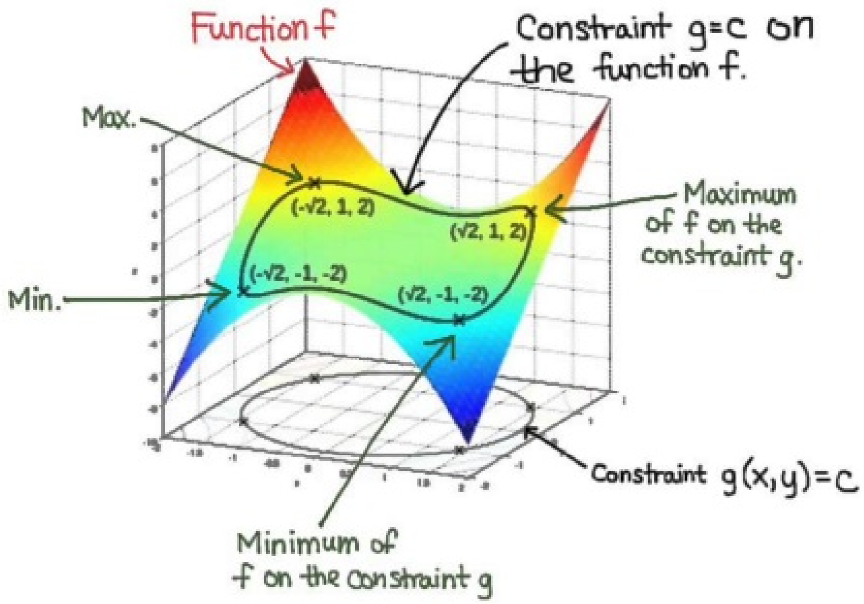
14.8: Lagrange Multipliers

PROBLEM: Maximize/minimize a general function $z = f(x, y)$ *subject to a constraint (or side condition) of the form* $g(x, y) = c$.



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METHOD OF LAGRANGE MULTIPLIERS: *To Maximize/minimize a general function $z = f(x, y)$ subject to a constraint of the form $g(x, y) = c$ (assuming that these extreme values exist):*

1. *Find all values x, y and λ (a Lagrange multiplier) s.t.*

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

(In other words $\nabla f \parallel \nabla g$)

and

$$g(x, y) = c$$

2. *Evaluate f at all points (x, y) that arise from the previous step. The largest of these values is the* max f*; the smallest is the* min f.

Rewrite the system

Find x, y and λ such that

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = c \end{cases}$$

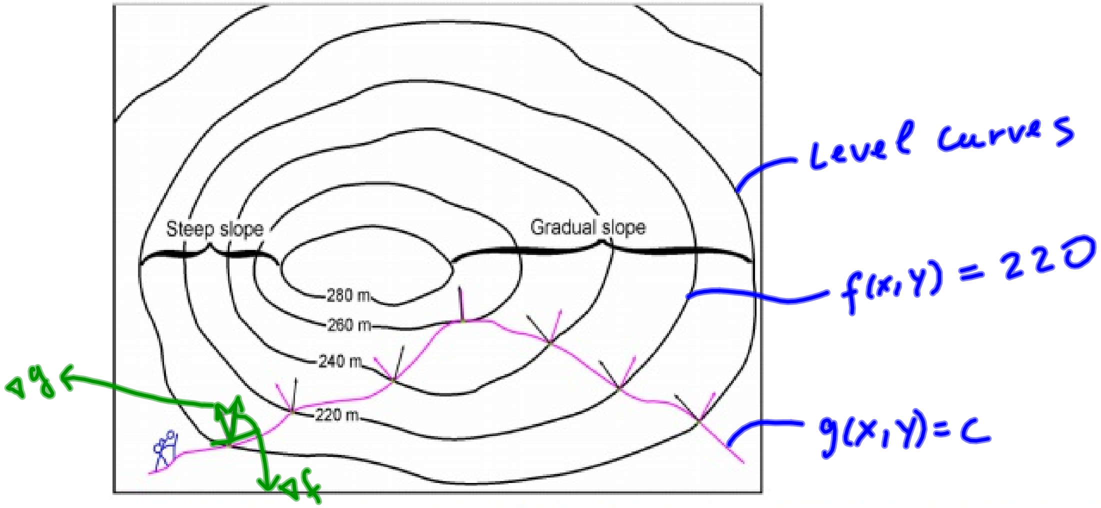
$\nabla f = \langle f_x, f_y \rangle$

$\nabla g = \langle g_x, g_y \rangle$

OR

in component form:.

$$\begin{cases} f_x(x, y) = \lambda g_x(x, y) \\ f_y(x, y) = \lambda g_y(x, y) \\ g(x, y) = c \end{cases}$$



https://math.stackexchange.com/questions/686538/how-to-explain-lagrange-multipliers-to- a - l a y - a u d i e n c e/686655

EXAMPLE 1. *Use Lagrange multipliers to find the maximum and minimum values of* $f(x, y) = x^2 + y^2$ *subject to* $\underbrace{x^4 + y^4}_{g(x,y)} = 1$. $g(x,y) = c$

Find x, y and λ such that

$$\begin{cases} \nabla f = \lambda \nabla g \\ x^4 + y^4 = 1 \end{cases} \Rightarrow \begin{cases} \langle \cancel{2}x, \cancel{2}y \rangle = \lambda \langle \cancel{4}^{2}x^3, \cancel{4}^{2}y^3 \rangle \\ x^4 + y^4 = 1 \end{cases} \Rightarrow$$

$$\begin{cases} x = 2\lambda x^3 \\ y = 2\lambda y^3 \\ x^4 + y^4 = 1 \end{cases} \Rightarrow \begin{cases} x - 2\lambda x^3 = 0 \\ y - 2\lambda y^3 = 0 \\ x^4 + y^4 = 1 \end{cases} \Rightarrow \begin{cases} x(1 - 2\lambda x^2) = 0 \\ y(1 - 2\lambda y^2) = 0 \\ x^4 + y^4 = 1 \end{cases} \Rightarrow$$

$$\begin{cases} x = 0 \\ y(1 - 2\lambda y^2) = 0 \\ x^4 + y^4 = 1 \end{cases} \quad \text{or} \quad \begin{cases} 1 - 2\lambda x^2 = 0 \\ y(1 - 2\lambda y^2) = 0 \\ x^4 + y^4 = 1 \end{cases}$$

First system, or:

$$\begin{cases} x = 0 \\ y = 0 \\ x^4 + y^4 = 1 \end{cases}$$

no solutions

$$\begin{cases} x = 0 \\ 1 - 2\lambda y^2 = 0 \\ x^4 + y^4 = 1 \end{cases}$$

$y = \pm 1$

$\lambda = \frac{1}{2y^2} = \frac{1}{2}$

$(x,y) = (0, \pm 1)$

Second system, or:

$$\begin{cases} 1 - 2\lambda x^2 = 0 \\ y = 0 \\ x^4 + y^4 = 1 \end{cases}$$

$\Downarrow$ similarly

$x = \pm 1$

$(x,y) = (\pm 1, 0)$

$$\begin{cases} 1 - 2\lambda x^2 = 0 \\ 1 - 2\lambda y^2 = 0 \\ x^4 + y^4 = 1 \end{cases}$$

$\frac{1}{2\lambda} = \boxed{x^2 = y^2}$

$x = \pm y$

$y^4 + y^4 = 1$

$2y^4 = 1$

$y^4 = \frac{1}{2}$

$y = \pm \frac{1}{\sqrt[4]{2}}$

$(x,y) = \underbrace{\left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}\right)}_{4 \text{ points.}}$

$f(x,y) = x^2 + y^2$

Caculate f at crit. point we found:

$f(0, \pm 1) = f(\pm 1, 0) = 1$

$f\left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}\right) = \left(\frac{1}{\sqrt[4]{2}}\right)^2 + \left(\frac{1}{\sqrt[4]{2}}\right)^2$

$= 2 \cdot \left(\frac{1}{\sqrt[4]{2}}\right)^2 = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$

Answer: $\max\limits_{x^4 + y^4 = 1} f(x,y) = \boxed{\sqrt{2}}$

$\min\limits_{x^4 + y^4 = 1} f(x,y) = \boxed{1}$