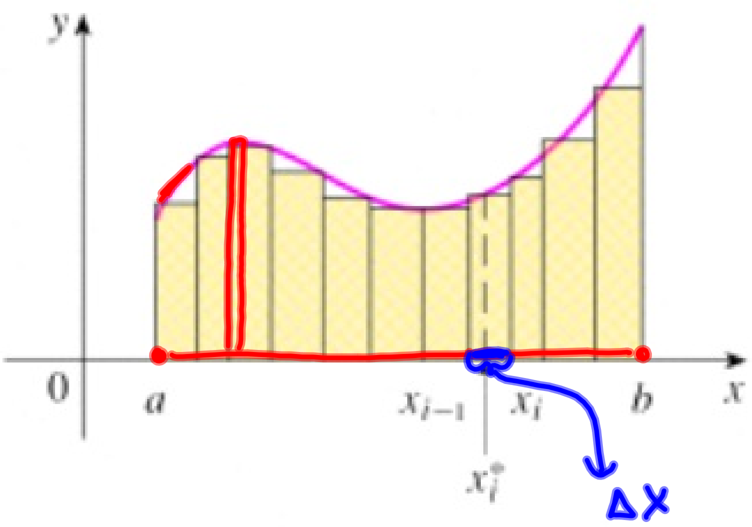
15.1: Double integrals over rectangles

Recall that a single definite integral can be interpreted as **area**:



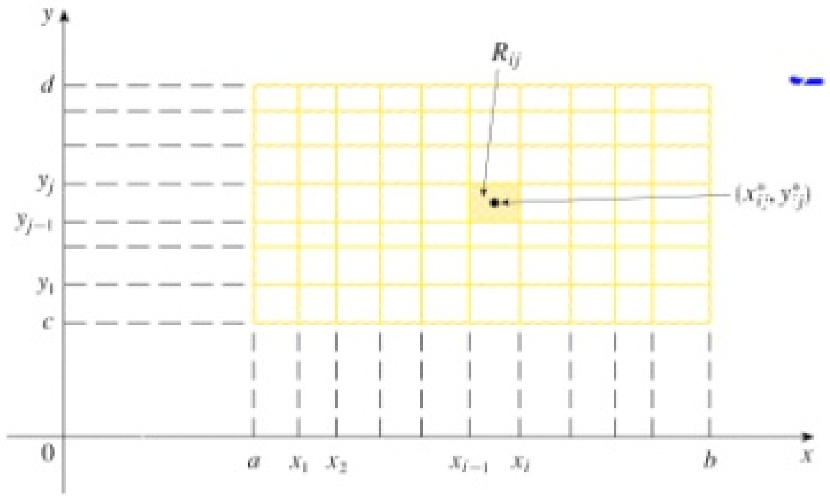
The exact area is also the definition of the definite integral:

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Problem: Assume that $f(x,y)$ is defined on a closed rectangle

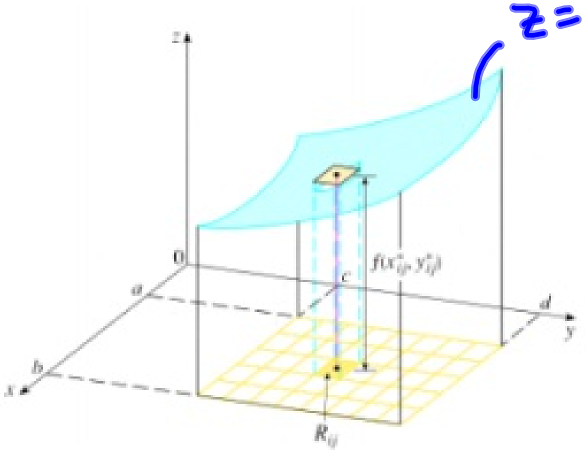
$R = [a,b] \times [b,c] = \left\{(x,y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\right\}$ and $f(x,y) \geq 0$ over R. Denote by S the part of the surface $z = f(x,y)$ over the rectangle R. What the volume of the region under S and above the xy-plane is?

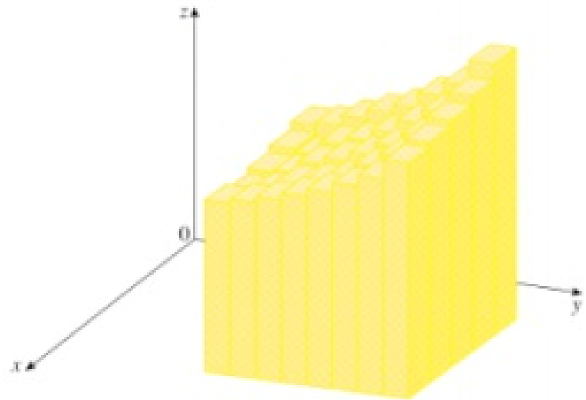
Solution: Approximate the volume. Divide up $a \leq x \leq b$ into n subintervals and divide up $c \leq y \leq d$ into m subintervals. From each of these smaller rectangles choose a point (x_i^*, y_j^*).



Over each of these smaller rectangles we will construct a box whose height is given by $f(x_i*, y_j^*)$.

$z = f(x,y)$





The volume is given by

$$\lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta x \Delta y$$

which is also the definition of a double integral

$$\iint_R f(x,y)\mathrm{d}A.$$

Another notation: $\iint_R f(x,y)\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y.$

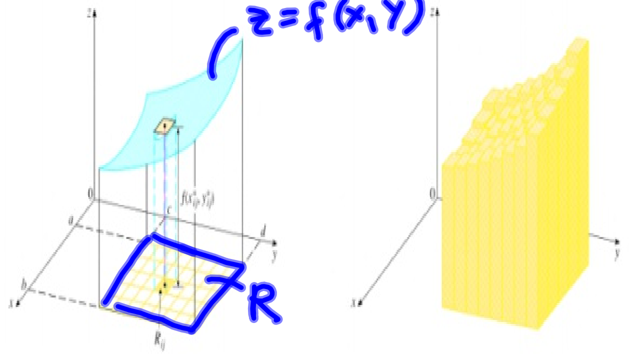
THEOREM 1. *If f is continuous on R then f is integrable over R.*

THEOREM 2. *If $f(x,y) \geq 0$ and f is continuous on the rectangle $R = [a,b] \times [c,d]$, then the volume V of the solid S that lies above R and under the graph of f, i.e.*

$$S = \{(x,y,z) \in \mathbb{R}^3 | (x,y) \in R, 0 \leq z \leq f(x,y), \cancel{(x,y) \in R}\},$$

is

$$V = \iint_R f(x,y)\,\mathrm{d}A.$$



EXAMPLE 3. *Evaluate the integral*

$$\iint_R \underbrace{4}_{=f(x,y)}\,\mathrm{d}A$$

where $R = [-1,0] \times [-3,3]$ by identifying it as a volume of a solid.

rectangle

$$\iint_R 4\,dA = \text{volume of the solid } S = \{(x,y,z) \in \mathbb{R}^3 | (x,y) \in R, 0 \leq z \leq 4\}$$

$$S = \{(x,y,z) \in \mathbb{R}^3 | -1 \leq x \leq 0,\ -3 \leq y \leq 3,\ 0 \leq z \leq 4\}$$

parallelopiped with base R and height 4

$$\iint_R 4\,dA = \text{Area}(R) \cdot 4 =$$

$$= (0-(-1)) \cdot (3-(-3)) \cdot 4 = 24 \text{ units}^3.$$