15.1 (ctd):Double Integrals over Rectangles

Iterated integrals

Suppose that $f(x, y)$ is integrable over the rectangle $R = [a, b] \times [b, c]$.

Partial integration of f with respect to x: $\int_a^b f(x, y)\,dx$

Partial integration of f with respect to y: $\int_c^d f(x, y)\,dy$

EXAMPLE 1.

$$\int_0^4 (x + 3y^2)\,dx = \left(\frac{x^2}{2} + 3y^2 x\right)\Big|_{x=0}^{4} = \frac{16}{2} + 3y^2 \cdot 4 - 0 = 8 + 12y^2$$

$$\int_1^4 e^{xy}\,dy = \frac{1}{x} e^{xy}\Big|_{y=1}^{4} = \frac{1}{x}\left(e^{4x} - e^{x}\right).$$

$\boxed{\textit{Iterated integrals:}}$

$$\text{a number} = \int_a^b \Big[\overbrace{\int_c^d f(x,y)\,\mathrm{d}y}^{\text{function of } x} \Big]\,\mathrm{d}x = \int_a^b \underbrace{\int_c^d} f(x,y)\,\mathrm{d}y\mathrm{d}x$$

and

$$\int_c^d \Big[\int_a^b f(x,y)\,\mathrm{d}x \Big]\,\mathrm{d}y = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 2. *Evaluate the integrals:*

iterated integrals

$$I_1 = \int_0^{\ln 2} \int_0^{\ln 5} e^{2x-y}\,\mathrm{d}y\mathrm{d}x, \quad I_2 = \int_0^{\ln 5} \int_0^{\ln 2} e^{2x-y}\,\mathrm{d}x\mathrm{d}y$$

$$I_1 = \int_0^{\ln 2} \Big[\int_0^{\ln 5} e^{2x-y}\,dy \Big] dx = \int_0^{\ln 2} \Big[\int_0^{\ln 5} e^{2x} \cdot e^{-y}\,dy \Big] dx =$$

$$= \int_0^{\ln 2} e^{2x} \Big[\underbrace{\int_0^{\ln 5} e^{-y}\,dy} \Big] dx = \int_0^{\ln 2} e^{2x} \Big[-e^{-y} \Big|_{y=0}^{\ln 5} \Big] dx =$$

$$= \int_0^{\ln 2} e^{2x} \underbrace{\left(e^0 - e^{-\ln 5}\right)} dx = \left(1 - \frac{1}{5}\right) \int_0^{\ln 2} e^{2x}\,dx$$

$$= \frac{4}{5} \frac{e^{2x}}{2} \Big|_0^{\ln 2} = \frac{2}{5}\left(e^{2\ln 2} - e^0\right)$$

$$= \frac{2}{5}(4-1) = \frac{6}{5}$$

In fact, $I_1 = \left(\int_0^{\ln 2} e^{2x}\,dx \right) \cdot \left(\int_0^{\ln 5} e^{-y}\,dy \right) = I_2.$

FUBINI's THEOREM: *If f is continuous on on the rectangle $R = [a, b] \times [c, d]$ then*

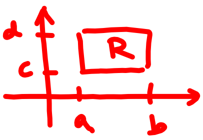
$$\underbrace{\iint_R f(x, y)\, \mathrm{d}A}_{\text{double}} = \underbrace{\int_a^b \int_c^d f(x, y)\, \mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x, y)\, \mathrm{d}x\mathrm{d}y.}_{\text{iterated integrals}}$$

EXAMPLE 3. *Make a conclusion from the Example 2 based on the Fubini Theorem.*

$$\iint_R e^{2x-y}\, dA = \frac{6}{5},$$

where $R = [0, \ln 2] \times [0, \ln 5]$

COROLLARY 4. *If g and h are continuous functions of one variable and $R = [a, b] \times [c, d]$ then*



$$\iint_R \underbrace{g(x)h(y)}_{f(x,y)}\,\mathrm{d}A = \left(\int_a^b g(x)\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right).$$

$f(x,y)$ is a product of two functions of one variable x and y, respectively.

f.ex.

$$f(x,y) = e^{x-2y} = \underbrace{e^{x}}_{g(x)} \cdot \underbrace{e^{-2y}}_{h(y)}$$

or

$$f(x,y) = \frac{x^3}{y^4} = \underbrace{x^3}_{g(x)} \cdot \underbrace{\frac{1}{y^4}}_{h(y)}$$

EXAMPLE 5. *Evaluate*

double

$$\iint_R x\cos(xy)\,\mathrm{d}A$$

where $R = \underbrace{[-\pi/2, \pi/2]}_{x} \times \underbrace{[1,5]}_{y}$ *and describe your result geometrically.*

$$\iint_R x\cos(xy)\,dA \overset{\text{Fubini Theorem}}{=} \int_1^5\left(\int_{-\pi/2}^{\pi/2} x\cos(xy)\,dx\right)dy$$

a better way

$$\int_{-\pi/2}^{\pi/2}\left(\int_1^5 x\cos(xy)\,dy\right)dx =$$

$$= \int_{-\pi/2}^{\pi/2} x\left(\int_1^5 \cos(xy)\,dy\right)dx =$$

$$= \int_{-\pi/2}^{\pi/2} \cancel{x}\left(\frac{1}{\cancel{x}}\sin(xy)\right)\Big|_{y=1}^{5} dx$$

$$= \int_{-\pi/2}^{\pi/2} (\sin(5x) - \sin x)\,dx$$

$$= \left(-\frac{\cos 5x}{5} + \cos x\right)\Big|_{-\pi/2}^{\pi/2} = 0.$$

EXAMPLE 6. *Express the volume of the solid S lying under the circular paraboloid $z = \overbrace{x^2 + y^2}^{f(x,y)}$ and above the rectangle $\underbrace{R = [-2, 2] \times [-3, 3]}_{\text{base}}$ using an iterated integral.*

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$f(x,y) = x^2 + y^2 \geq 0$

By Th. 2 we have

$$V = \iint_R f(x,y)\, dA = \iint_R (x^2+y^2)\, dA =$$

$$= \int_{-2}^{2} \int_{-3}^{3} (x^2+y^2)\, dy\, dx$$