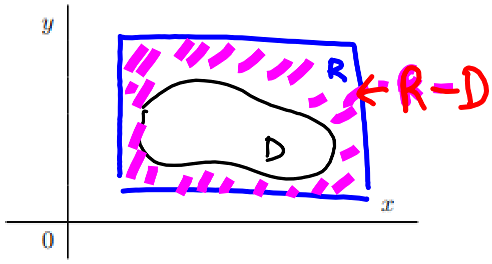
15.2: Double Integrals over General Regions

`All functions below are continuous on their domains.`

Let D be a bounded region enclosed in a rectangular region R. We define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D. \end{cases}$$



If F is integrable over R, then we say F is *integrable* over D and we define **the double integral of f over D** by

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A$$

$$= \iint_R F(x,y)\,dA - \underbrace{\iint_{R-D} F(x,y)\,dA}_{=0}$$

FACT: If $f(x,y) \geq 0$ and f is continuous on the region D then the volume V of the solid S that lies above D and under the graph of f, i.e.

$$S = \left\{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x,y), (x,y) \in D\right\},$$

is

$$V = \iint_D f(x,y)\,\mathrm{d}A.$$

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EXAMPLE 1. *Evaluate the integral*

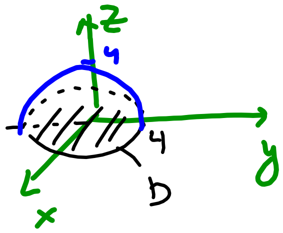
$$\iint_D \sqrt{16 - x^2 - y^2}\, \mathrm{d}A$$

where $D = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 16\}$ by identifying it as a volume of a solid.

base

Lid $z = \sqrt{16 - x^2 - y^2}$

upper semisphere

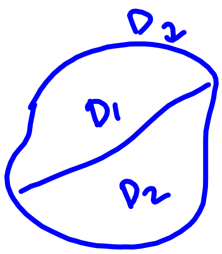


$\iint_D \sqrt{16 - x^2 - y^2}$ = volume of the solid semi sphere with radius 4

$= \frac{1}{2} \cdot \frac{4}{3}\pi \cdot 4^3 = \frac{128\pi}{3}$ unit^3

Properties of double integrals:

- If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps their boundaries then
$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A.$$



- If α and β are real numbers then

$$\iint_D (\alpha f(x,y) + \beta g(x,y))\,\mathrm{d}A = \alpha \iint_D f(x,y)\,\mathrm{d}A + \beta \iint_D g(x,y)\,\mathrm{d}A.$$

- If we integrate the constant function $f(x,y) = 1$ over D, we get **area** of D:

$$\iint_D 1\,\mathrm{d}A = A(D) \cdot 1$$

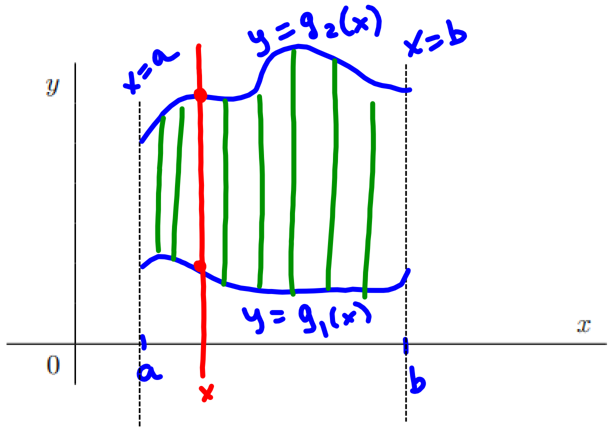
EXAMPLE 2. *If* $D = \{(x,y) | \ x^2 + y^2 \leq 25\}$ *then*

$$\iint_D \mathrm{d}A = \text{area of } D = 25\pi$$

Computation of double integral:

A plain region of **TYPE I**:

$$D = \{(x, y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}\,.$$



THEOREM 3. *If D is a region of type I such that $D = \{(x, y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}$ then*

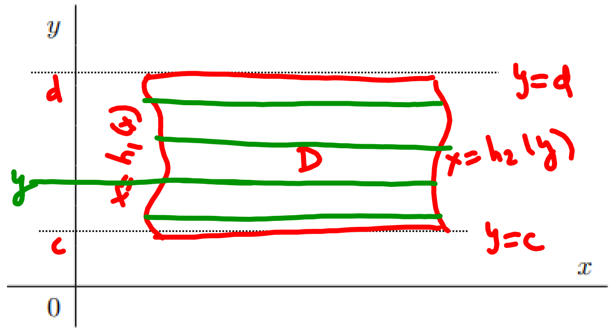
upper curve .

$$\iint_D f(x, y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,\mathrm{d}y\mathrm{d}x.$$

Lower curve

A plain region of **TYPE II**:

$$D = \{(x, y)|\ c \le y \le d, h_1(y) \le x \le h_2(y)\}\,.$$

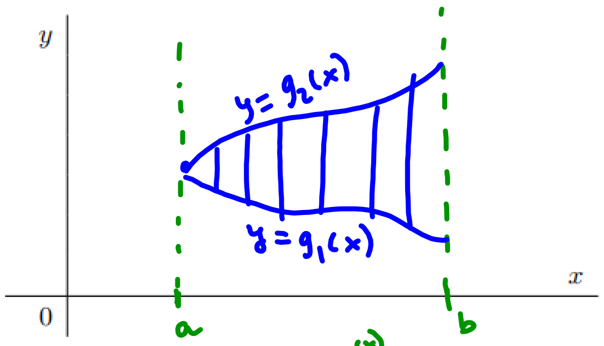


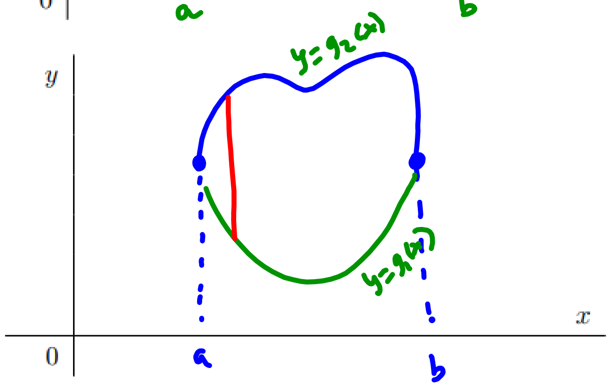
THEOREM 4. *If D is a region of type II s.t. $D = \{(x, y)|\ c \le y \le d, h_1(y) \le x \le h_2(y)\}$ then*

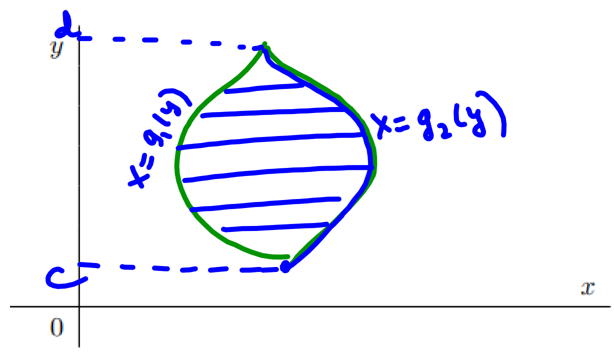
$$\iint_D f(x, y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

← right curve

← left curve



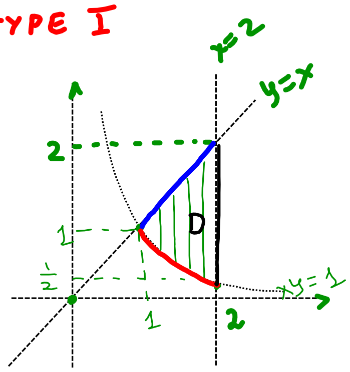




EXAMPLE 5. *Evaluate* $I = \iint_D 30x^2y\,dA$, *where* D *is the region bounded by the lines* $x = 2, y = x$ *and the hyperbola* $xy = 1$ *in two different ways (i.e. considering* D *as a type I and then as a type II region).*

TYPE I

$$D = \{(x, y) \mid 1 \le x \le 2,\ \tfrac{1}{x} \le y \le x\}$$



Lower curve $y = g_1(x) = \frac{1}{x}$

Upper curve $y = g_2(x) = x$

$$I = \int_1^2 \left(\int_{\frac{1}{x}}^{x} 30x^2 y\,dy \right) dx =$$

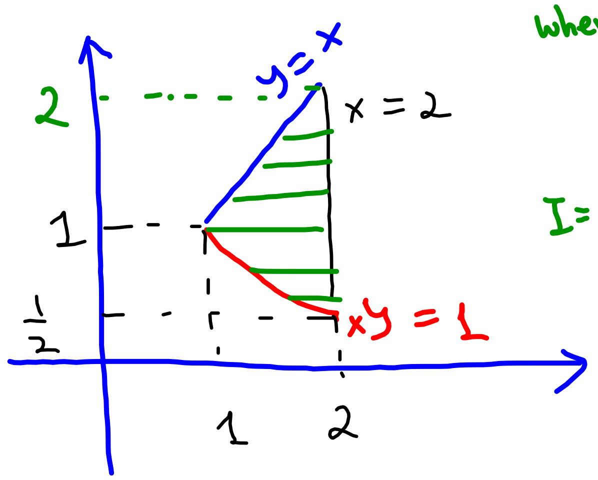
$$= 30 \int_1^2 x^2 \left(\int_{\frac{1}{x}}^{x} y\,dy \right) dx = \cancel{30}^{15} \int_1^2 x^2 \left(\frac{y^2}{\cancel{2}} \Big|_{y=\frac{1}{x}}^{x} \right) dx =$$

$$= 15 \int_1^2 x^2 \left(x^2 - \frac{1}{x^2} \right) dx = 15 \int_1^2 (x^4 - 1)\,dx = 15 \left(\frac{x^5}{5} - x \right) \Big|_1^2 =$$

$$= 15 \left(\frac{32}{5} - 2 - \frac{1}{5} + 1 \right) =$$

$$= 15 \left(\frac{31}{5} - 1 \right) = \frac{\cancel{15}^{3}(31-5)}{\cancel{5}_1} = 78$$

TYPE 2



$$D = \left\{ (x,y) \,\middle|\, \tfrac{1}{2} \le y \le 2 \,,\; h_1(y) \le x \le \underbrace{2}_{\text{right curve}} \right\}$$

where $x = \underbrace{h_1(y)}_{\text{left curve}} = \begin{cases} \frac{1}{y}, & \frac{1}{2} \le y \le 1 \\ y & 1 \le y \le 2 \end{cases}$

$$I = \iint_D 30x^2y \, dA =$$

$$= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{y}}^{2} 30x^2y \, dx \, dy + \int_{1}^{2} \int_{y}^{2} 30x^2y \, dx\,dy$$

$= \dots$

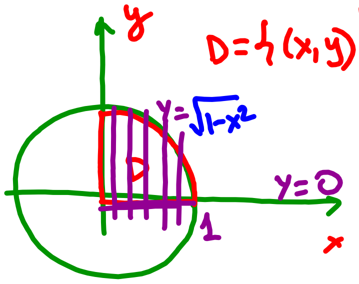
EXAMPLE 6. *Find the volume of the solid bounded by the cylinder* $x^2 + y^2 = 1$ *and the planes* $\cancel{x = 0}$, $y = z$, $z = 0$ *in the first octant.*

$x \geq 0$, $y \geq 0$, $z \geq 0$

$z = f(x,y)$ lid

$z = y$

$$V = \iint_D \underbrace{f(x,y)}_{y}\, dA$$



$D = \{(x,y) \mid x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$

$$V = \iint_D y\, dA = \dots$$

$$= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} y\, dy \right) dx =$$

$x^2 + y^2 = 1$

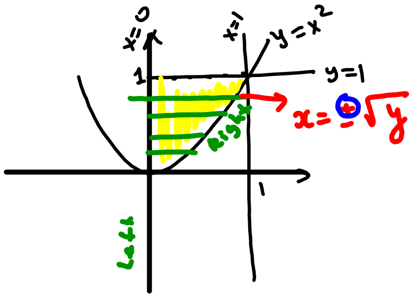
$y = \pm\sqrt{1-x^2}$

$$= \int_0^1 \frac{y^2}{2} \Bigg|_{y=0}^{\sqrt{1-x^2}} dx = \int_0^1 \frac{1-x^2}{2}\, dx = \left(\frac{x}{2} - \frac{x^3}{6} \right) \Bigg|_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3} \text{ units}^3$$

EXAMPLE 7. *Evaluate the following iterated integral by reversing the order of integration:*

$$I = \int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy dx = \iint_D x^3 \sin(y^3)\, dA,$$

where $D = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le 1\}$



$$I = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy =$$

$$= \int_0^1 \sin y^3 \left(\int_0^{\sqrt{y}} x^3\, dx \right) dy = \int_0^1 \sin y^3 \left(\frac{x^4}{4} \Big|_{x=0}^{\sqrt{y}} \right) dy$$

$$= \int_0^1 \sin(y^3) \left(\frac{y^2}{4} - \frac{0}{4} \right) dy = \frac{1}{4} \int_0^1 y^2 \sin(y^3)\, dy = \ldots$$

u-sub

$u = y^3 \Rightarrow du = 3y^2\, dy$