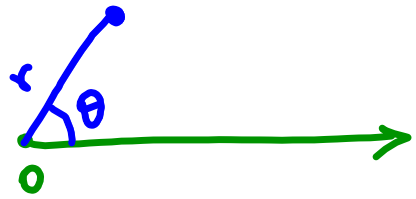
15.3: Double Integrals in Polar Coordinates



The **polar coordinate system** consists of:

- the **pole** (or origin) labeled O;
- the **polar axis** which is a ray starting at O (usually drawn horizontally to the right);

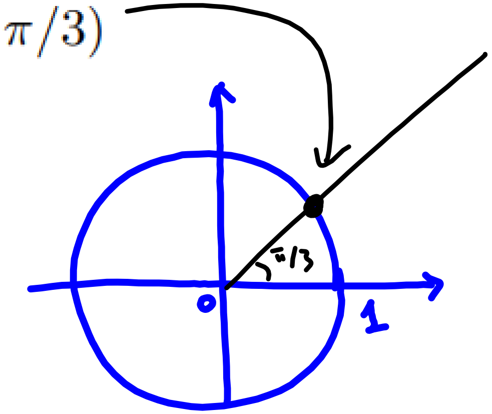
The **polar coordinates** (r, θ) of a point P:

- θ is the angle between the polar axis and the line OP (the angle is positive if measured in counter-clockwise direction from the polar axis);
- r is the distance from O to P.

EXAMPLE 1. *Plot the points whose polar coordinates are given:*

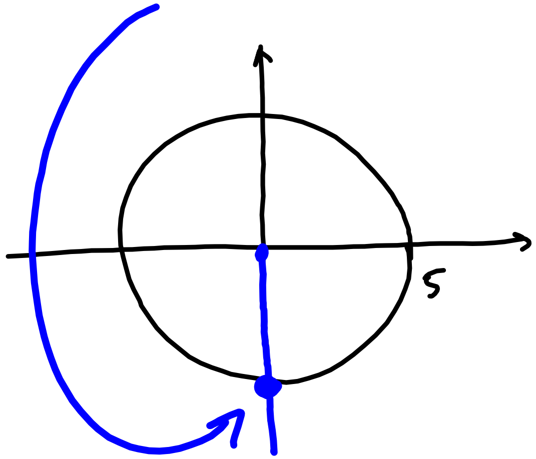
(a) $(1, \pi/3)$

$r = 1$
$\theta = \frac{\pi}{3}$

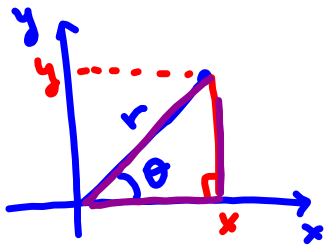


(b) $(5, -\pi/2)$.

$r = 5$
$\theta = -\pi/2$



The connection between polar and Cartesian coordinates:



$r = \sqrt{x^2+y^2}$

$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}} \qquad \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2+y^2 \qquad \tan\theta = \frac{y}{x}$$

REMARK 2. In converting from the Cartesian to polar coordinates we must choose θ so that the point (r, θ) lies in the correct quadrant.

EXAMPLE 3. *What curve is represented by the following polar equation*

(a) $r = 12$

$r = \sqrt{x^2+y^2} = 12$

$x^2 + y^2 = 144$

circle

(b) $\theta = \frac{\pi}{3}$

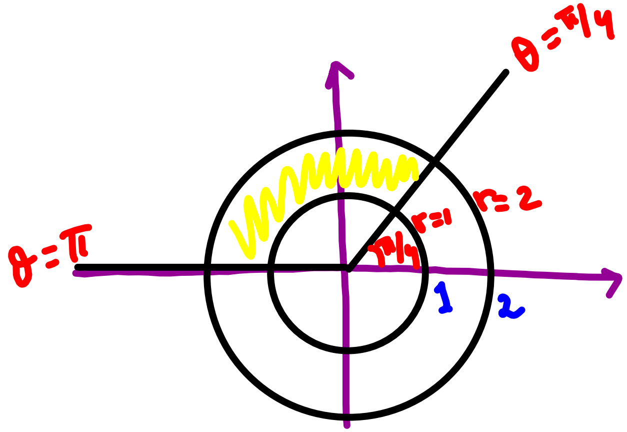
$\tan\theta = \frac{y}{x}$

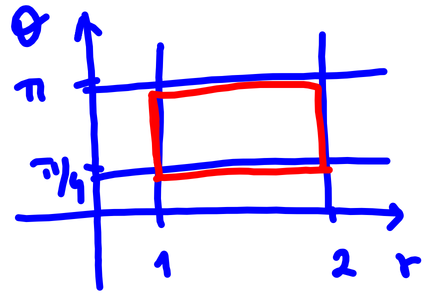
$\tan\frac{\pi}{3} = \sqrt{3} = \frac{y}{x}$

$y = x\sqrt{3}$

line

EXAMPLE 4. *Sketch the region in the Cartesian plane consisting of points whose polar coordinates satisfy the following conditions:* $1 \le r \le 2, \quad \pi/4 \le \theta \le \pi.$





EXAMPLE 5. *Find a polar equation for the curve represented by the given Cartesian equation:*

(a) $x^2 + y^2 = 2by$ (b is a constant)

$x = r\cos\theta$, $y = r\sin\theta$

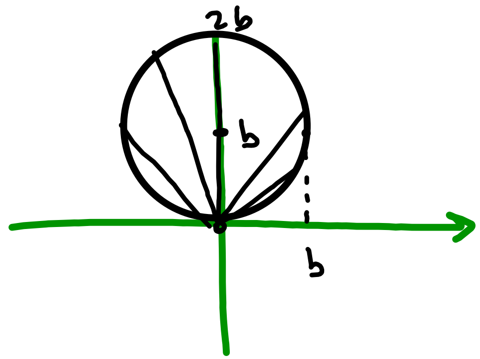
$r^2 = 2br\sin\theta$

$\boxed{r = 2b\sin\theta}$

$x^2 + y^2 = 2by$

$x^2 + (y^2 - 2by + b^2) - b^2 = 0$

$x^2 + (y-b)^2 = b^2$



(b) $(x-a)^2 + y^2 = a^2$

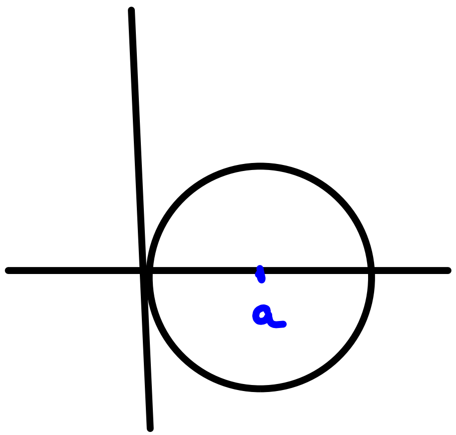
$x^2 - 2ax + \cancel{a^2} + y^2 = \cancel{a^2}$

$x^2 + y^2 = 2ax$

$\boxed{x = r\cos\theta \quad , \quad y = r\sin\theta}$

$r^2 = 2ar\cos\theta$

$\boxed{r = 2a\cos\theta}$

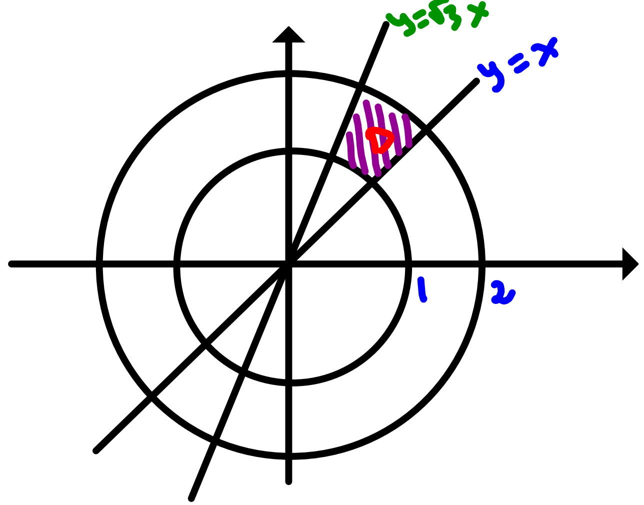


Using polar coordinates to evaluate double integrals

EXAMPLE 6. *Evaluate*

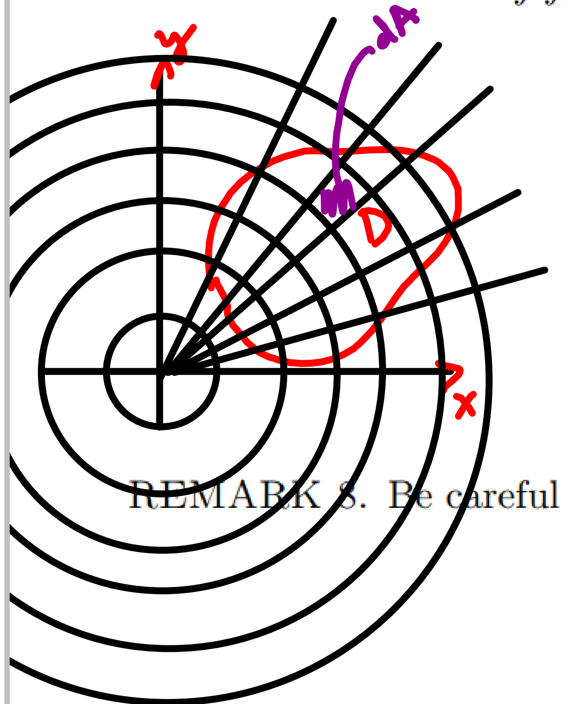
$$I = \iint_D \arctan \frac{y}{x} \, \mathrm{d}A$$

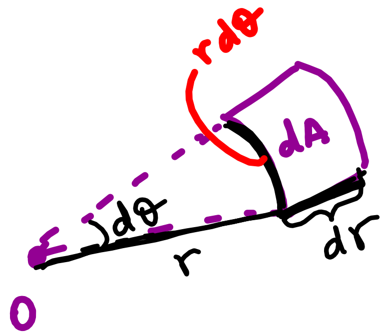
where $D = \left\{ (x, y) \middle| \ 1 \leq x^2 + y^2 \leq 4, x \leq y \leq \sqrt{3}x, x \geq 0 \right\}.$



THEOREM 7. *Change to polar coordinates in a double integral: Let f be a continuous on the region D. Denote by D^* the region representing D in the polar coordinates (r, θ). Then*

$$\iint_D f(x, y)\,\mathrm{d}A = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}r\mathrm{d}\theta.$$





$dA = r\,d\theta \cdot dr$

REMARK 8. Be careful not to forget the additional factor r on the right side of the formula.

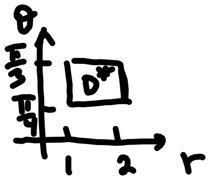
Solution of Example 6:

Evaluate $I = \displaystyle\iint_D \arctan\frac{y}{x}\,\mathrm{d}A$, where $D = \left\{(x,y) \mid 1 \le x^2+y^2 \le 4, x \le y \le \sqrt{3}x, x \ge 0\right\}$.

$$D = \left\{ (x,y) \mid 1 \le \sqrt{x^2+y^2} \le 2,\ 1 \le \frac{y}{x} \le \sqrt{3},\ x \ge 0 \right\}$$

use polar coordinates

$$D^* = \left\{ (r,\theta) \mid 1 \le r \le 2,\ 1 \le \tan\theta \le \sqrt{3},\ \underbrace{r\cos\theta \ge 0}_{-\pi/2 \le \theta \le \pi/2} \right\}$$



$$D^* = \left\{ (r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3} \right\}$$

$$I = \iint_D \arctan\frac{y}{x}\,dA = \iint_{D^*} \arctan(\tan\theta)\, r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/3}\int_1^2 \theta\, r\,dr\,d\theta = \int_{\pi/4}^{\pi/3} \theta\,d\theta \cdot \int_1^2 r\,dr = \frac{1}{2}\theta^2\Big|_{\pi/4}^{\pi/3} \frac{1}{2}r^2\Big|_1^2$$

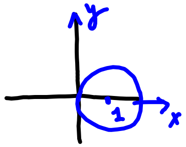
$$= \frac{1}{4}\left(\frac{\pi^2}{9} - \frac{\pi^2}{16}\right)(4-1) = \ldots$$

EXAMPLE 9. *Find the volume of the solid that lies under the paraboloid* $z = x^2 + y^2$, *above the xy-plane and inside the cylinder* $x^2 + y^2 = 2x$.

$$V = \iint_D f(x,y)\, dA$$

$f(x,y) = x^2 + y^2$

$$D = \{(x,y) \mid \underbrace{x^2 + y^2 \le 2x}_{(x-1)^2 + y^2 \le 1}\}$$



$$V = \iint_D (x^2 + y^2)\, dA$$

Using polar coordinates:

$$D^* = \{(r,\theta) \mid r^2 \le 2r\cos\theta\}$$

$$= \{(r,\theta) \mid 0 \le r \le \underbrace{2\cos\theta}_{\ge 0},\ -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}\}$$

$$V = \iint_{D^*} r^2\, r\, dr\, d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} \left(\int_0^{2\cos\theta} r^3\, dr \right) d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{r^4}{4} \Big|_{r=0}^{2\cos\theta} \right) d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} \frac{16\cos^4\theta}{4}\, d\theta = 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = 4\int_{-\pi/2}^{\pi/2} (\cos^2\theta)^2\, d\theta$$

$$(\cos^2\theta)^2 = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right)$$

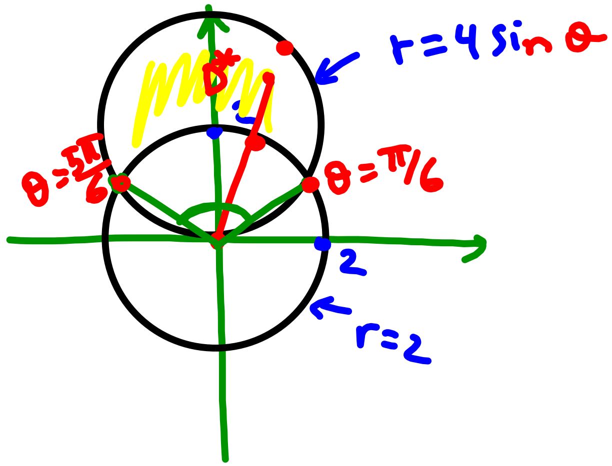
$$= \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)$$

$$V = \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta \right) d\theta = \dots$$

EXAMPLE 10. *Find the area of the region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$.*

$$A = \iint_D dA = \iint_{D^*} r\,dr\,d\theta =$$

$$= \int_{\pi/6}^{5\pi/6} \int_2^{4\sin\theta} r\,dr\,d\theta$$



To find Bound for θ:

$$\begin{cases} r = 4\sin\theta \\ r = 2 \end{cases} \Rightarrow 4\sin\theta = 2$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$= \int_{\pi/6}^{5\pi/6} \frac{r^2}{2}\Big|_{r=2}^{4\sin\theta} d\theta = \int_{\frac{\pi}{6}}^{5\pi/6} \left(8\sin^2\theta - 2\right) d\theta$$

$$= \int_{\pi/6}^{5\pi/6} 4(1-\cos 2\theta) - 2 \; d\theta = \ldots$$