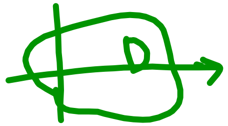
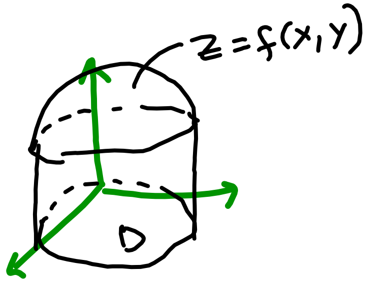
15.4: Applications of double integral

- **Area:** $A(D) = \iint_D \mathrm{d}A$



- **Volume:** $V(D) = \iint_D f(x,y)\,\mathrm{d}A$, where f is nonnegative on D.



- **Total Mass** m of the lamina with variable (nonhomogeneous) density $\rho(x,y)$, where the function ρ is continuous on D:

$$m = \iint_D \rho(x,y)\,\mathrm{d}A.$$

$m = \rho \cdot A(D)$

Total mass $\approx \sum_k \sum_i \rho(x_i, y_k)\,\Delta A_{ik}$

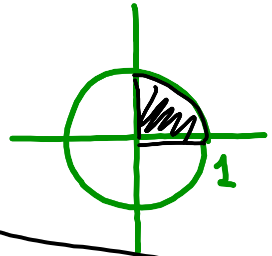
$m_{ik} = \rho(x_i, y_k) \cdot \Delta A_{ik}$

- **Total charge** Q: If an electric charge is distributed over a region D and the charge density (units of charge per unit area) is given by $\sigma(x,y)$ at a point (x,y) in D, then the total charge Q is given by

$$Q = \iint_D \sigma(x,y)\,\mathrm{d}A.$$

EXAMPLE 1. *Charge is distributed over the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant so that the charge density at (x, y) is $\sigma(x, y) = x^2 + y^2$, measured in coulombs per square meter (C/m^2). Find the total charge.*

$$\text{Total charge} = \iint_D \sigma(x,y)\, dA$$



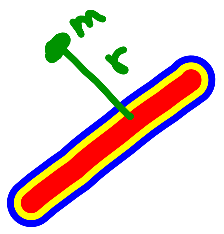
$$= \iint_D (x^2+y^2)\, dA =$$

where $D = \{(x,y) \mid x^2+y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$

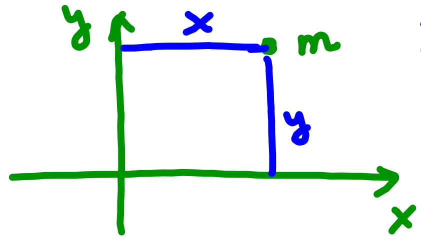
$D^* = \{(r,\theta) \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi/2\}$

$$\iint_{D^*} \underbrace{r^2}_{x^2+y^2} \cdot \underbrace{r\,dr\,d\theta}_{dA} = \int_0^{\pi/2} \left(\int_0^1 r^2 r\, dr \right) d\theta$$

$$= \frac{\pi}{2} \left. \frac{r^4}{4} \right|_0^1 = \boxed{\frac{\pi}{8}}\ C/m^2$$



moment = m · r



$M_x = m \cdot y$

$M_y = m\, x$

- **Moment** of the lamina with variable (nonhomogeneous) density $\rho(x, y)$ that occupies the region D **about the x-axis:**

  ← distance to the x-axis

  $$M_x = \iint_D y\, \underbrace{\rho(x, y)\, \mathrm{d}A}_{\text{mass}}$$

  Moment of the lamina **about the y-axis:**

  $$M_y = \iint_D x\rho(x, y)\, \mathrm{d}A$$

- **Center of mass** , $(\bar{x}, \bar{y})$, of the lamina with variable (nonhomogeneous) density $\rho(x, y)$ that occupies the region D is defined so that

  $$\boxed{m\bar{x} = M_y, \qquad m\bar{y} = M_x.}$$

  These yield

  $$\bar{x} = \frac{\iint_D x\rho(x, y)\, \mathrm{d}A}{m}, \qquad \bar{y} = \frac{\iint_D y\rho(x, y)\, \mathrm{d}A}{m},$$

  where $m = \iint_D \rho(x, y)\, \mathrm{d}A$.

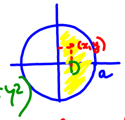
REMARK 2. The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Thus, the lamina balances horizontally when supported as its center of mass.

EXAMPLE 3. *Find the center of mass of the lamina that occupies the region*

$$D = \{(x, y) : x^2 + y^2 \le a^2, x \ge 0\}$$

if the density at any point is proportional to the square of its distance from the origin.

$$m = \iint_D \rho(x,y)\, dA, \text{ where}$$



$$\rho(x,y) = K\left(\sqrt{x^2+y^2}\right)^2 = K(x^2+y^2),$$

where K is a coefficient.

$$\bar{x} = \frac{\iint_D x \cancel{K}(x^2+y^2)\, dA}{\iint_D \cancel{K}(x^2+y^2)\, dA}$$

see calculations below

$$\bar{y} = \frac{\iint_D y \cancel{K}(x^2+y^2)\, dA}{\iint_D \cancel{K}(x^2+y^2)\, dA} = 0$$

because the density is symmetric w.r.t. the x-axis and the lamina is symmetric w.r.t. the x-axis. So, the center of mass should be on the x-axis.

Calculations:

First find mass of the lamina

$$m = \iint_D \rho(x,y)\, dA = K\iint_D (x^2+y^2)\, dA = K\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^a r^2\, r\, dr\, d\theta$$

$$= K \underbrace{\int_{-\pi/2}^{\pi/2} d\theta}_{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)}\ \frac{r^4}{4}\Big|_0^a = \frac{K\cdot\pi a^4}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x,y)\, dA = \frac{K}{m}\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^a r\cos\theta\, r^2\, r\, dr\, d\theta$$

$$= \frac{K}{m}\left(\int_{-\frac{\pi}{2}}^{\pi/2}\cos\theta\, d\theta\right)\left(\int_0^a r^4\, dr\right) = \frac{K}{m}\sin\theta\Big|_{-\pi/2}^{\pi/2}\cdot\frac{r^5}{5}\Big|_0^a$$

$$= \frac{\cancel{K}}{\frac{\cancel{K\pi a^4}}{4}}\underbrace{(1-(-1))}_{2}\frac{a^{\cancel{5}}}{5} = \frac{8a}{5\pi}$$

Final answer $\boxed{\left(\frac{8a}{5\pi}, 0\right)}$