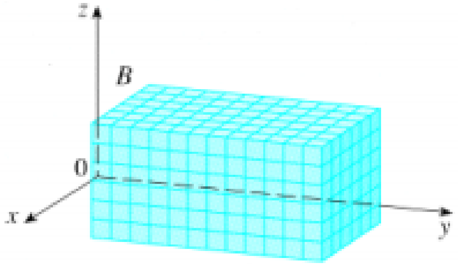
15.6: Triple Integrals

Mass problem: *Given a solid object, that occupies the region B in $\mathbb{R}^3$, with density $\rho(x, y, z)$. Find the mass of the object.*

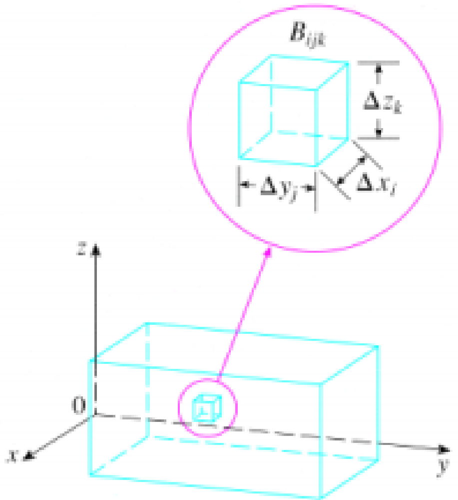
Solution: Let B be a rectangular box:



$$B = \{(x, y, z) | a \le x \le b, c \le y \le d, r \le z \le s\}$$

$\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$

Partition in sub-boxes:



$$m_{ijk} = \rho(x_i^*, y_j^*, z_k^*) \Delta V_{ijk}$$

$$\|P\| = \max \sqrt{\Delta x_i^2 + \Delta y_j^2 + \Delta z_k^2}$$

$$m = \lim_{\|P\| \to 0} \sum_i \sum_j \sum_k \rho(x_i^*, y_j^*, z_k^*) \Delta V_{ijk}$$

$$m = \iiint_B \rho(x, y, z) \, dV$$

FUBINI's THEOREM: *If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$ then*

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\mathrm{d}z$$

and there are 5 other possible orders in which we can integrate.

EXAMPLE 1. *Let $B = [0,1] \times [-1,3] \times [0,3]$. Evaluate*

$$I = \iiint_B xye^{yz}\,\mathrm{d}V = \int_{-1}^{3}\left(\int_0^3\left(\underbrace{\int_0^1 xye^{yz}\,dx}\right)dz\right)dy =$$

$$= \int_{-1}^{3}\int_0^3 ye^{yz}\left(\frac{x^2}{2}\Big|_{x=0}^{1}\right)^{=\frac{1}{2}} dz\,dy = \frac{1}{2}\int_{-1}^{3}\int_0^3 ye^{yz}\,dz\,dy =$$

$$= \frac{1}{2}\int_{-1}^{3} \cancel{y}\cdot\frac{1}{\cancel{y}}e^{yz}\Big|_{z=0}^{3} dy = \frac{1}{2}\int_{-1}^{3}(e^{3y} - \underset{1}{e^0})\,dy = \ldots$$

$$= \left(\frac{e^{3y}}{3} - y\right)\Big|_{-1}^{3}\cdot\frac{1}{2} = \frac{1}{2}\left[\frac{e^9}{3} - 3 - \left(\frac{e^{-3}}{3} + 1\right)\right]$$

$$= \frac{1}{2}\left[\frac{e^9 - e^{-3}}{3} - 4\right]$$

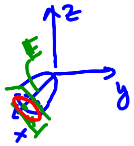
FACT: The volume of the solid E is given by the integral,

$$V = \iiint_E dV.$$

FACT: The mass of the solid E with variable density $\rho(x, y, z)$ is given by the integral,

$$m = \iiint_E \rho(x,y,z)\,dV.$$

EXAMPLE 2. *Find the mass of the solid bounded by $x = y^2 + z^2$ and the plane $x = 4$ if the density function is $\rho(x, y, z) = \sqrt{y^2 + z^2}$.*



$$m = \iiint_E \rho(x,y,z)\,dV = \iiint_E \sqrt{y^2+z^2}\,dV,$$

where $E = \{(x,y,z) \mid y^2+z^2 \le x \le 4,\ (y,z) \in D\}$,

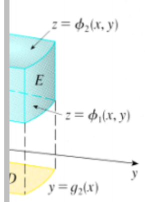
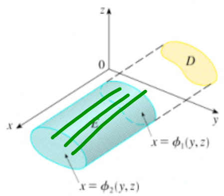
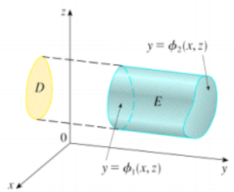
where D is bounded by the line of intersection between paraboloid $x = y^2+z^2$ and plane $x = 4$

$$\left.\begin{array}{l} x = y^2+z^2 \\ x = 4 \end{array}\right\} \Rightarrow y^2+z^2 = 4$$

This means $D = \{(y,z) \mid y^2+z^2 \le 4\}$

projection of the solid E onto the yz-plane.

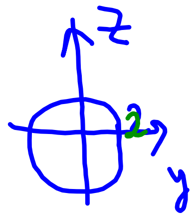
Table 1: **Triple integrals over a general bounded region** E

TYPE I:	A solid region of **TYPE II**: our case	A solid region of **TYPE III**:
$\in D, \phi_1(x,y) \le z \le \phi_2(x,y)\}$ projection of E onto the xy- ... on	$E = \{(x,y,z) \mid (y,z) \in D, \phi_1(y,z) \le x \le \phi_2(y,z)\}$ where D is the projection of E onto the yz-plane.	$E = \{(x,y,z) \mid (x,z) \in D, \phi_1(x,z) \le y \le \phi_2(x,z)\}$ where D is the projection of E onto the xz-plane.
	 A type 2 region	 A type 3 region
$= \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\right] dA$	$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{\phi_1(y,z)}^{\phi_2(y,z)} f(x,y,z)\,dx\right] dA$ ($dA = dy\,dz$)	$\iiint_E f(x,y,z)\,dV =$

a triple integral it is wise to draw **two** diagrams: one of the solid region E and one of its projection on the corresponding coordinate plane.

$$m = \iint_D \left(\int_{y^2+z^2}^{4} \sqrt{y^2+z^2}\, dx \right) dA =$$

$$= \iint_D \sqrt{y^2+z^2} \left(\int_{y^2+z^2}^{4} dx \right) dA$$



$$= \iint_D \sqrt{y^2+z^2} \left(4 - (y^2+z^2) \right) dA =$$

$$=$$

use polar coordinates

$y = r\cos\theta$, $z = r\sin\theta$

$D^* = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

$y^2+z^2 = r^2$

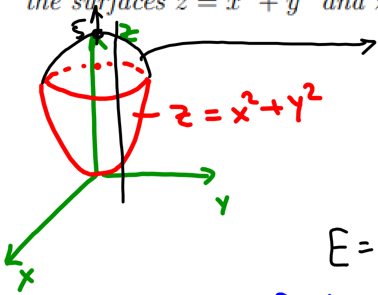
$dA = r\,dr\,d\theta$

$$m = \iint_{D^*} r(4-r^2)\, r\, dr\, d\theta = \ldots$$

$$= \int_0^{2\pi} \int_0^2 r(4-r^2)\, r\, dr\, d\theta = 2\pi \int_0^2 (4r^2 - r^4)\, dr$$

$$= 2\pi \left(\frac{4r^3}{3} - \frac{r^5}{5} \right) \Big|_0^2 = \ldots$$

EXAMPLE 3. *Use a triple integral to find the volume of the solid bounded by the surfaces $z = x^2 + y^2$ and $z = 5 - 4x^2 - 4y^2$.*



$z = 5 - 4(x^2+y^2)$

$z = x^2+y^2$

$$V = \iiint_E dV \,, \quad \text{where}$$

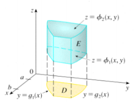
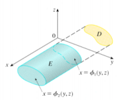
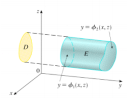
$$E = \{ (x,y,z) \mid x^2+y^2 \le z \le 5-4(x^2+y^2), (x,y) \in D \}$$

where D is a projection of E onto the xy-plane.
The boundary of D is the line of intersection of the given paraboloids:

$$x^2+y^2 = 5-4(x^2+y^2)$$
$$5(x^2+y^2) = 5$$
$$x^2+y^2 = 1$$

So, $D = \{ (x,y) \mid x^2+y^2 \le 1 \}$

Table 1: **Triple integrals over a general bounded region** E

our case A solid region of **TYPE I**: $E = \{(x,y,z) \mid (x,y) \in D, \phi_1(x,y) \le z \le \phi_2(x,y)\}$ where D is the projection of E onto the xy-plane.	A solid region of **TYPE II**: $E = \{(x,y,z) \mid (y,z) \in D, \phi_1(y,z) \le x \le \phi_2(y,z)\}$ where D is the projection of E onto the yz-plane.	A solid region of **TYPE III**: $E = \{(x,y,z) \mid (x,z) \in D, \phi_1(x,z) \le y \le \phi_2(x,z)\}$ where D is the projection of E onto the xz-plane.
A type 1 solid region 	 A type 2 region	 A type 3 region
$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz \right] dA$	$\iiint_E f(x,y,z)\,dV = \iint \left[\int f(x,y,z)\,dx \right] dA$	$\iiint_E f(x,y,z)\,dV =$

When we set up a triple integral it is wise to draw **two** diagrams: one of the solid region E and one of its projection on the corresponding coordinate plane.

$$V = \iint_D \left(\int_{x^2+y^2}^{5-4(x^2+y^2)} dz \right) dA =$$

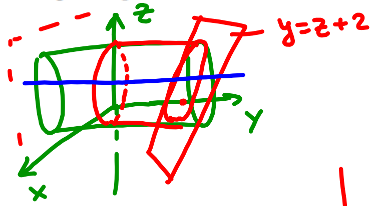
$$V = \iint_D \left(5-4(x^2+y^2) - (x^2+y^2) \right) dA$$

$$V = \iint_D \left(5 - 5(x^2+y^2) \right) dA$$

$$V = 5 \underbrace{\iint_D dA}_{\text{area of } D} - 5 \underbrace{\iint_D (x^2+y^2)\,dA}_{\text{use polar coord.}} = 5 \cdot \pi \cdot 1^2 - 5 \int_0^{2\pi}\int_0^1 r^2 \, r\,dr\,d\theta$$

$$= \boxed{\frac{5\pi}{2}} \text{ unit}^3$$

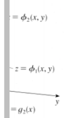
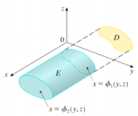
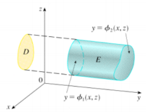
EXAMPLE 4. *Use a triple integral to find the volume of the solid bounded by the elliptic cylinder* $4x^2 + z^2 = 4$ *and the planes* $y = 0$ *and* $y = z + 2$.



$$E = \{(x,y,z) \mid 0 \le y \le z+2,\ (x,z) \in D\}$$

where $D = \{(x,z) \mid 4x^2 + z^2 \le 4\}$

Table 1: **Triple integrals over a general bounded region** E

I:	A solid region of **TYPE II**:	A solid region of **TYPE III**:
$x,y) \le z \le \phi_2(x,y)\}$ on of E onto the xy-	$E = \{(x,y,z) \mid (y,z) \in D, \phi_1(y,z) \le x \le \phi_2(y,z)\}$ where D is the projection of E onto the yz-plane.	$E = \{(x,y,z) \mid (x,z) \in D, \phi_1(x,z) \le y \le \phi_2(x,z)\}$ where D is the projection of E onto the xz-plane.
	 A type 2 region	 A type 3 region
$\left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\right] dA$	$\iiint_E f(x,y,z)\,dV = \iint \left[\int f(x,y,z)\,dx\right] dA$	$\iiint_E f(x,y,z)\,dV =$

e integral it is wise to draw **two** diagrams: one of the solid region E and one of its projection on the corresponding coordinate plane.

$$V = \iiint_E dV = \iint_D \left(\int_0^{z+2} dy\right) \underbrace{dA}_{= dx\,dz} =$$

$$= \iint_D (z + 2 - 0)\, dA =$$

$$= \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} (2+z)\, dz\, dx$$

$$= \int_{-1}^{1} \left(\frac{z^2}{2} + 2z\right)\Bigg|_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} dx$$

$$V = \int_{-1}^{1} \left(0 + 2\cdot\left(2\sqrt{1-x^2} - \left(-2\sqrt{1-x^2}\right)\right)\right) dx$$

$$V = 8\int_{-1}^{1} \sqrt{1-x^2}\, dx = 8\cdot(\text{Area of half unit disk})$$

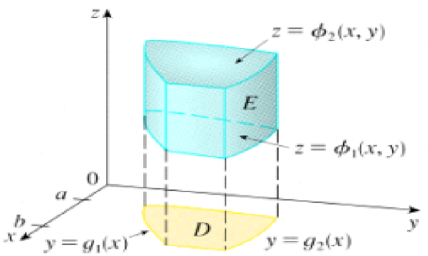
$$= 8\,\frac{\pi}{2} = \boxed{4\pi}\ \text{unit}^3$$

Table 1: **Triple integrals over a general bounded region E**

A solid region of **TYPE I**:

$E = \{(x,y,z) | (x,y) \in D, \phi_1(x,y) \le z \le \phi_2(x,y)\}$ where D is the projection of E onto the xy-plane.

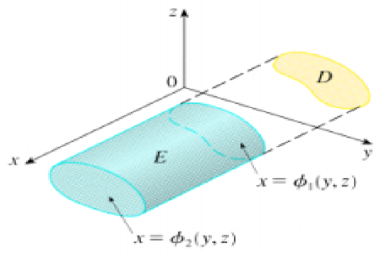
A type 1 solid region



$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,\mathrm{d}z\right]\underbrace{\mathrm{d}A}_{dxdy}$$

A solid region of **TYPE II**:

$E = \{(x,y,z) | (y,z) \in D, \phi_1(y,z) \le x \le \phi_2(y,z)\}$ where D is the projection of E onto the yz-plane.

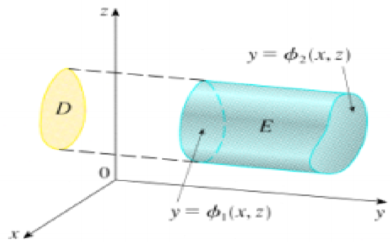


A type 2 region

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{\varphi_1(y,z)}^{\varphi_2(y,z)} f(x,y,z)\,\mathrm{d}x\right]\underbrace{\mathrm{d}A}_{dydz}$$

A solid region of **TYPE III**:

$E = \{(x,y,z) | (x,z) \in D, \phi_1(x,z) \le y \le \phi_2(x,z)\}$ where D is the projection of E onto the xz-plane.



A type 3 region

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{\varphi_1(x,z)}^{\varphi_2(x,z)} f\,dy\right]\underbrace{dA}_{dxdz}$$

When we set up a triple integral it is wise to draw **two** diagrams: one of the solid region E and one of its projection on the corresponding coordinate plane.