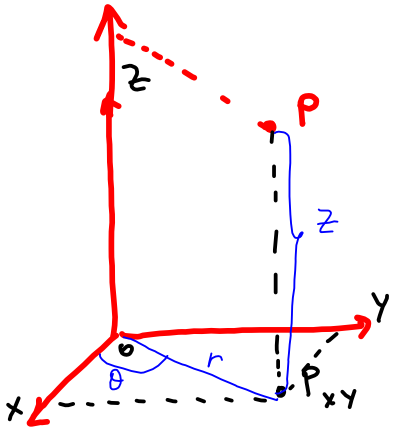
15.7: Triple integrals in cylindrical coordinates

- **Cylindrical coordinates:**



$P(x, y, z) \in \mathbb{R}^3$
In the cylindrical coordinates P is represented by the ordered triple (r, θ, z), where r, θ are the polar coordinates of P_{xy} and z is the directed distance from the xy-plane to P:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

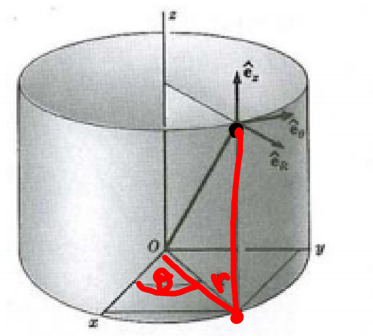
where

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z.$$

REMARK 1. The cylindrical coordinates

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ r &\geq 0, \quad 0 \leq \theta \leq 2\pi \end{aligned}}$$

are useful in problems that involve *symmetry about the z-axis.*

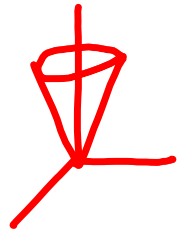


(a) Cylindrical

https://i.stack.imgur.com/FgSBF.jpg

EXAMPLE 2. *Find an equation in cylindrical coordinates for the cone*

$$z = \sqrt{x^2 + y^2}$$



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$z = \sqrt{r^2}$

$\boxed{z = r}$

THEOREM 3. *Let $f(x,y,z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in cylindrical coordinates. Then*

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(r\cos\theta, r\sin\theta, z)\,\mathrm{d}V^*,$$

where

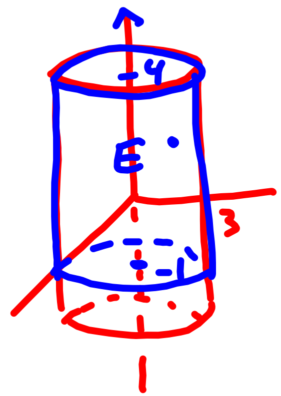
$$\boxed{\mathrm{d}V^* = r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta.}$$

EXAMPLE 4. *The density at any point of the solid E,*

$$E = \{(x, y, z) : x^2 + y^2 \le 9, -1 \le z \le 4\},$$

& symmetry

equals to its distance from the axis of E. Find the mass of E.

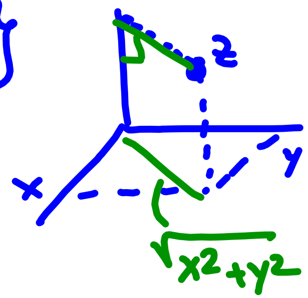


$$m(E) = \iiint_E \rho(x,y,z)\, dV = \iiint_E \sqrt{x^2+y^2}\, dV$$

use cylindrical coordinates

$$E^* = \{(r, \theta, z) \mid r^2 \le 9,\ -1 \le z \le 4,\ r \ge 0,\ 0 \le \theta \le 2\pi\}$$

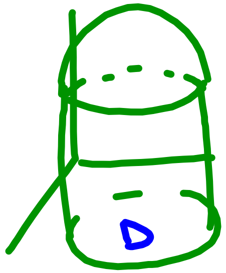
$$E^* = \{(r, \theta, z) \mid 0 \le r \le 3,\ -1 \le z \le 4,\ 0 \le \theta \le 2\pi\}$$



$$m(E) = \iiint_{E^*} \sqrt{r^2}\, dV^* = \int_0^{2\pi}\int_{-1}^{4}\int_0^3 r\, r\, dr\, dz\, d\theta = \ldots$$

$$= 5 \cdot 2\pi \cdot \int_0^3 r^2\, dr = 10\pi \left.\frac{r^3}{3}\right|_0^3 = \boxed{90\pi}$$

REMARK 5. If E is a solid region of type I, i.e.



$$E = \{(x, y, z) | (x, y) \in D, \phi_1(x, y) \leq z \leq \phi_2(x, y)\},$$

where D is the projection of E onto the xy-plane then, as we know,

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x, y, z)\, \mathrm{d}z \right] \mathrm{d}A.$$

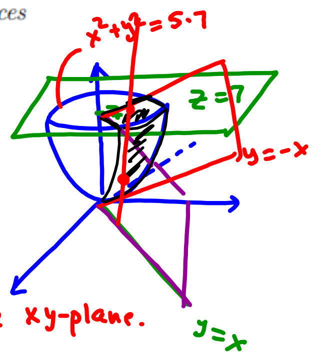
Passing to cylindrical coordinates here we actually have to replace D by its image D^* in polar coordinates and $\mathrm{d}z\, \mathrm{d}A$ by $r\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta$.

EXAMPLE 6. *Find the volume of the solid E bounded by the surfaces*

$y = x, \quad y = -x, \quad x^2 + y^2 = 5z, \quad z = 7$

$z = \frac{x^2+y^2}{5}$

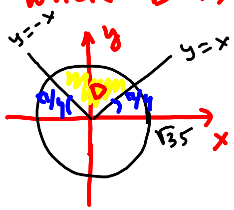
so that $y \geq 0$.



$$V(E) = \iiint_E dV = \iint_D \left(\int_{\frac{x^2+y^2}{5}}^{7} dz \right) dA$$

where D is a projection of E onto the xy-plane.

To find it: $\left.\begin{array}{l} x^2+y^2=5z \\ z=7 \end{array}\right\} \Rightarrow x^2+y^2=35$



$D = \{(x,y) \mid x^2+y^2 \leq 35, -y \leq x \leq y, y \geq 0\}$

$$V(E) = \iint_D \left(7 - \frac{x^2+y^2}{5}\right) dA =$$

polar coordinates

$$= \iint_{D^*} \left(7 - \frac{r^2}{5}\right) r\,dr\,d\theta =$$

$$= \int_{\pi/4}^{3\pi/4} \int_0^{\sqrt{35}} \left(7 - \frac{r^2}{5}\right) r\,dr\,d\theta$$

$$= \frac{\pi}{2}\left(\frac{7r^2}{2} - \frac{r^4}{20}\right)\Bigg|_0^{\sqrt{35}} = \frac{\pi}{2}\left(\frac{7\cdot 35}{2} - \frac{35^2}{20}\right)$$

$$= \frac{\pi}{2}\cdot 7\cdot 35\underbrace{\left(\frac{1}{2} - \frac{5}{20}\right)}_{\frac{1}{4}} = \boxed{\frac{245\pi}{8} \text{ unit}^3}$$