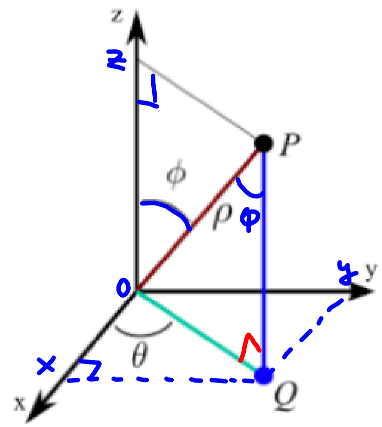
15.8: Triple integrals in spherical coordinates

- **Spherical coordinates** of P is the ordered triple (ρ, θ, ϕ) where $|OP| = \rho$, $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.



$|OP| = \rho$

$x = |OQ| \cos\theta$

$y = |OQ| \sin\theta$

$z = \rho\cos\varphi$

$|OQ| = \rho\sin\varphi$

The spherical coordinates

$$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \\ \rho \geq 0, &\quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi \end{aligned}}$$

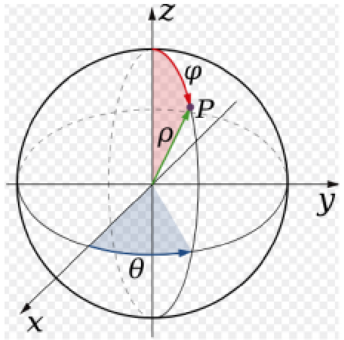
are especially useful in problems where there is *symmetry about the origin*.

Note that

$x^2+y^2+z^2 = \rho^2\sin^2\varphi + \rho^2\cos^2\varphi$

$= \rho^2(\sin^2\varphi + \cos^2\varphi) = \rho^2$

$$\boxed{x^2+y^2+z^2 = \rho^2}$$

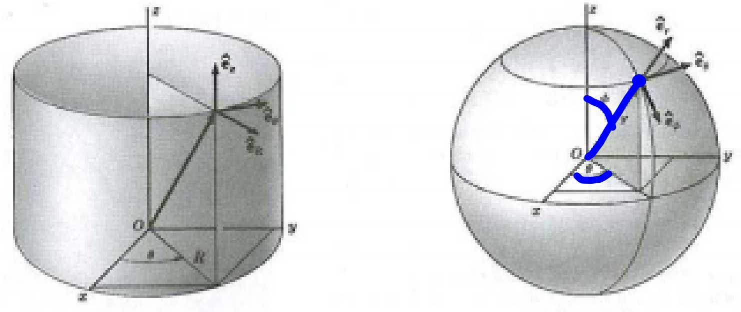


https://commons.wikimedia.org/wiki/

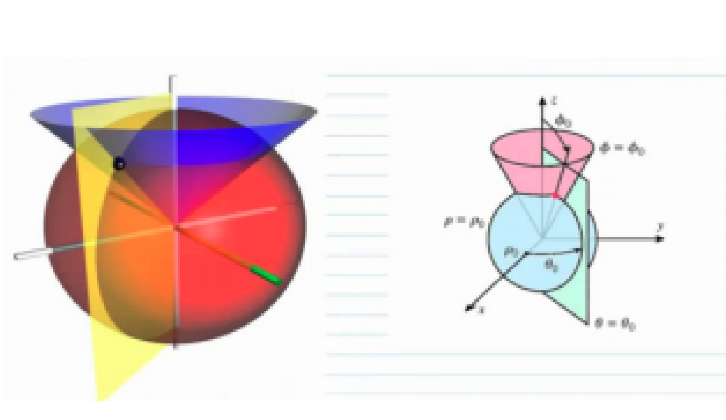
$x^2+y^2 = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta$

$= \rho^2\sin^2\varphi\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}$

$$\boxed{x^2+y^2 = \rho^2\sin^2\varphi}$$



(a) Cylindrical (b) Spherical



https://i.stack.imgur.com/FgSBF.jpg *https://www.youtube.com/watch?v=Q-RUZIboBeE*

EXAMPLE 1. *Find equation in spherical coordinates for the following surfaces.*

(a) $x^2 + y^2 + z^2 = 16$

$\rho^2 = 16$

$\rho = \pm 4$, $\rho \geq 0$

$\boxed{\rho = 4}$

(b) $z = \sqrt{x^2 + y^2}$

cone

$z = \rho \cos \varphi$

$x^2 + y^2 = \rho^2 \sin^2 \varphi$

$\Rightarrow \rho \cos \varphi = \sqrt{\rho^2 \sin^2 \varphi}$

$\rho \cos \varphi = |\rho \sin \varphi|$

But $\rho > 0$

and $\sin \varphi > 0$

$(0 \leq \varphi \leq \pi)$

$\cancel{\rho} \cos \varphi = \cancel{\rho} \sin \varphi$

$\tan \varphi = 1$

$\boxed{\varphi = \frac{\pi}{4}}$

(c) $z = \sqrt{3x^2 + 3y^2}$ $\Rightarrow z = \sqrt{3} \sqrt{x^2 + y^2}$

$\rho \cos \varphi = \sqrt{3}\, \rho \sin \varphi$

$\tan \varphi = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \boxed{\varphi = \frac{\pi}{6}}$

(d) $x = y$

$\rho \sin \varphi \cos \theta = \rho \sin \varphi \sin \theta$

$\cos \theta = \sin \theta$

$\tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$

•Triple integrals in spherical coordinates

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$
$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

$$(x, y, z) \quad \rightarrow \quad (r, \theta, z) \quad \rightarrow \quad (\rho, \theta, \phi)$$

$$x = r\cos\theta \qquad z = \rho\cos\phi$$
$$y = r\sin\theta \qquad \boxed{r = \rho\sin\phi}$$
$$z = z \qquad \theta = \theta$$

$$dV = dx\,dy\,dz = r\,dr\,dz\,d\theta = r\,\rho\,d\rho\,d\theta\,d\phi = \rho\sin\phi\,\rho\,d\rho\,d\theta\,d\phi$$
$$= \underbrace{\rho^2\sin\phi}\,d\rho\,d\theta\,d\phi$$

THEOREM 2. *Let $f(x,y,z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in spherical coordinates. Then*

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\underline{\rho^2\sin\phi}\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

EXAMPLE 3. *Evaluate* $I = \iiint_E e^{\sqrt{(x^2+y^2+z^2)^3}}\,\mathrm{d}V$ *where* $E = \{(x,y,z) : 9 \le \underbrace{x^2+y^2+z^2}_{\rho^2} \le 16\}$.

Apply spherical coordinates:

$$E^* = \{(\rho, \theta, \varphi) \mid \underbrace{9 \le \rho^2 \le 16,\ \rho \ge 0}, 0 \le \theta \le 2\pi, 0 \le \varphi \le \pi\}$$

$$E^* = \{(\rho, \theta, \varphi) \mid 3 \le \rho \le 4, 0 \le \theta \le 2\pi, 0 \le \varphi \le \pi\}$$

$$I \overset{Th.2}{=} \iiint_{E^*} e^{\sqrt{(\rho^2)^3}} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi =$$

$$= \left(\int_0^{\pi} \sin\varphi \, d\varphi\right) \cdot \left(\int_0^{2\pi} d\theta\right) \cdot \left(\int_3^4 e^{\rho^3} \rho^2 \, d\rho\right) =$$

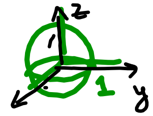
EXAMPLE 4. *Write the integral* $\iiint_E f(x,y,z)\,dV$ *in spherical coordinates where*

(a) $E = \{(x,y,z) : x^2+y^2+z^2 \le 1, y \ge 0, z \ge 0\}$.

$E^* = \{(\rho,\theta,\varphi) \mid \boxed{\rho^2 \le 1},\ \rho \underbrace{\sin\varphi}_{\ge 0} \sin\theta \ge 0,\ \rho\cos\varphi \ge 0,\ \boxed{\rho \ge 0},\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$

$E^* = \{(\rho,\theta,\varphi) \mid 0 \le \rho \le 1,\ \sin\theta \ge 0,\ \cos\varphi \ge 0,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$

$E^* = \{(\rho,\theta,\varphi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le \pi,\ 0 \le \varphi \le \pi/2\}$

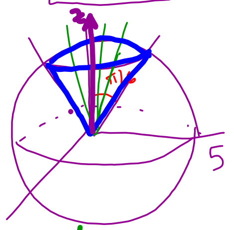


$I = \int_0^1 \int_0^\pi \int_0^{\pi/2} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\, d\varphi\, d\theta\, d\rho$

(b) *E is the ice cream cone-shaped solid, which is cut from the sphere of radius 5 by the cone* $\boxed{\phi = \pi/6.}$

First way (using picture)

$E^* = \{(\rho,\theta,\varphi) \mid 0 \le \rho \le 5,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi/6\}$



sphere

$x^2+y^2+z^2 = 25$

$z = \pm\sqrt{25-x^2-y^2}$

cone $\varphi = \pi/6$ (see ex. 1)

$z = \sqrt{3(x^2+y^2)}$

Second way

$E = \{(x,y,z) \mid \sqrt{3(x^2+y^2)} \le z \le \sqrt{25-x^2-y^2}\}$

$E^* = \{(\rho,\theta,\varphi) \mid \underbrace{\sqrt{3\rho^2\sin^2\varphi} \le \rho\cos\varphi}\ \le \sqrt{25-\rho^2\sin^2\varphi},\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi,\ \rho \ge 0\}$

$E^* = \{(\rho,\theta,\varphi) \mid \sqrt{3}\rho\sin\varphi \le \rho\cos\varphi \le \sqrt{25-\rho^2\sin^2\varphi},\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi,\ \rho \ge 0\}$

$\sqrt{3}\sin\varphi \le \cos\varphi$

$\tan\varphi \le \frac{1}{\sqrt{3}}$

$0 \le \varphi \le \pi/6$

$\rho^2\cos^2\varphi \le 25 - \rho^2\sin^2\varphi$

$\rho^2\cos^2\varphi + \rho^2\sin^2\varphi \le 25$

$\rho^2 \le 25$

$0 \le \rho \le 5$

$I = \int_0^{2\pi}\int_0^{\pi/6}\int_0^5 f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

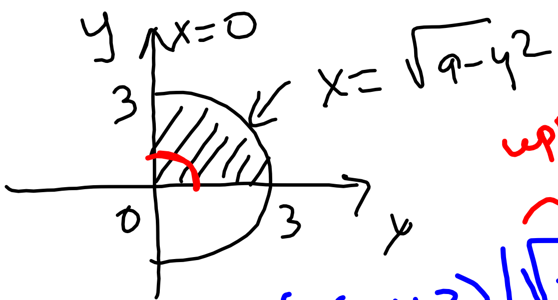
Title: Mar 29-1:45 PM (6 of 8)

EXAMPLE 5. *Evaluate the integral by changing to spherical coordinates:*

$$I = \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} \overbrace{(x^2+y^2+z^2)}^{\rho^2}\, dz\, dx\, dy.$$

$$= \iint_D \left[\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\, dz \right] dA,$$

where $D = \{(x,y) \mid 0 \le y \le 3,\ 0 \le x \le \sqrt{9-y^2}\}$



$$E = \{(x,y,z) \mid \overbrace{\sqrt{x^2+y^2}}^{\text{upper cone}} \le z \le \overbrace{\sqrt{18-x^2-y^2}}^{\text{hemisphere with radius } \sqrt{18}}\}$$

$$E^* = \{(\rho, \theta, \phi) \mid 0 \le \rho \le \sqrt{18},\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le \pi/4\}$$

$$I = \int_0^{\sqrt{18}} \int_0^{\pi/2} \int_0^{\pi/4} \rho^2 \, \underbrace{\rho^2 \sin\varphi \, d\varphi \, d\theta \, d\rho}_{dV^*}$$

$$= \left(\int_0^{\sqrt{18}} \rho^4 \, d\rho \right) \left(\int_0^{\pi/2} d\theta \right) \left(\int_0^{\pi/4} \sin\varphi \, d\varphi \right) =$$

$$= \frac{\pi}{2} \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^{3\sqrt{2}} \rho^4 \, d\rho$$

$$= \frac{\pi}{2} \left(-\cos\varphi \right) \Big|_0^{\pi/4} \frac{\rho^5}{5} \Big|_0^{3\sqrt{2}} = \ldots$$