15.9: Change Of Variables In Double Integral

Examples of a change of variables:

- substitution rule
$$\int_a^b f(g(x))\underbrace{g'(x)\,\mathrm{d}x} = \int_\alpha^\beta f(u)\,\underbrace{\mathrm{d}u}.$$
- conversion to polar coordinates:
$$\iint_D f(x,y)\,\underbrace{\mathrm{d}A} = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,\underbrace{r\,\mathrm{d}r\,\mathrm{d}\theta}_{dA^*}.$$
$$r = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|$$
- conversion to cylindrical coordinates:
$$\iiint_E f(x,y,z)\,\underbrace{\mathrm{d}V} = \iiint_{E^*} f(r\cos\theta, r\sin\theta, z)\,\underbrace{r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta}_{dV^*}$$
$$r = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|$$
- conversion to spherical coordinates:
$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\underbrace{\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi}_{dV^*}.$$
$$\rho^2\sin\phi = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right|$$

We call the equations that define the change of variables a **transformation:**

$$x = x(u,v), \quad y = y(u,v).$$

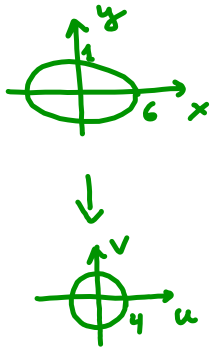
EXAMPLE 1. *Determine the new region that we get by applying the transformation $x = 3u, y = v/2$ to the region* $D = \left\{(x,y) \mid \frac{x^2}{36} + y^2 \leq 1\right\}$.

$$\frac{x^2}{36} + y^2 \le 1$$

$$\frac{(3u)^2}{36} + \left(\frac{v}{2}\right)^2 \le 1$$

$$\frac{9u^2}{36} + \frac{v^2}{4} \le 1$$

$$\frac{u^2}{4} + \frac{v^2}{4} \le 1 \Rightarrow u^2 + v^2 \le 4$$



DEFINITION 2. *The* **Jacobian** *of the transformation* $x = x(u, v), y = y(u, v)$ *is*

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

EXAMPLE 3. *Compute the Jacobian of the transformation* $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta) = r(\cos^2\theta + \sin^2\theta) = \underline{\underline{r}}$$

Change of variables for a double integral:

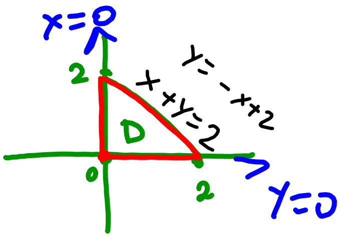
abs value

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v.$$

EXAMPLE 4. *Evaluate*

$$\iint_D e^{\frac{y-x}{y+x}}\,\mathrm{d}A$$

where D is triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$.



Substitution: $u = y-x$ (1)
$v = y+x$ (2)

inverse Jacobian

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

(1)+(2) $2y = u+v$

$y = \frac{u+v}{2}$

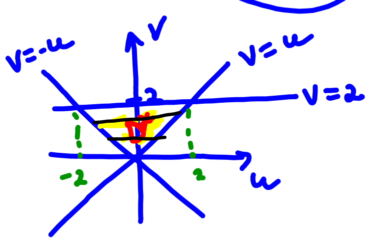
(2)-(1) $v-u = 2x$

$\boxed{x = \frac{v-u}{2}}$

$\boxed{\partial D}$ $x=0$, $y=0$, $x+y=2$ $\boxed{\begin{matrix} u=y-x \\ v=y+x \end{matrix}}$ $\boxed{\partial D^*}$ $\frac{v-u}{2}=0$, $\frac{u+v}{2}=0$, $v=2$

∂D^* OR

$\boxed{\begin{matrix} v=u \\ v=-u \\ v=2 \end{matrix}}$



$$\iint_D e^{\frac{y-x}{y+x}}\,dA = \iint_{D^*} e^{\frac{u}{v}} \left|-\frac{1}{2}\right| du\,dv =$$

$$= \frac{1}{2}\int_0^2 \left(\int_{-v}^{v} e^{\frac{u}{v}}\,du \right) dv =$$

$$= \frac{1}{2}\int_0^2 v e^{\frac{u}{v}} \Big|_{u=-v}^{v} dv =$$

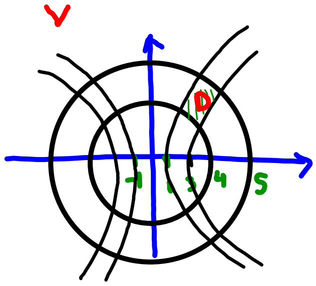
$$= \frac{1}{2}\int_0^2 v\left(e^{1} - e^{-1}\right) dv =$$

$$= \frac{1}{2}\left(e - \frac{1}{e}\right)\int_0^2 v\,dv = e - \frac{1}{e}.$$

EXAMPLE 5. *Find mass of a lamina that occupies the region*

$$D = \{(x,y) | 16 \le \underbrace{x^2+y^2}_{u} \le 25, \quad 1 \le \underbrace{x^2-y^2}_{v} \le 9, \quad \overbrace{x \ge 0, y \ge 0}^{\text{1st quadrant}}\}$$

with density $\rho(x,y) = 8xy$.



$$m = \iint_D \rho(x,y)\, dA = \iint_D 8xy\, dA$$

Change of variables:

$u = x^2+y^2$, $v = x^2-y^2$.

Then $D^* = \{(u,v) \mid 16 \le u \le 25,\ 1 \le v \le 9\}$

Inverse Jacobian $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} = -4xy - 4xy$

$= -8xy$

Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1} = -\frac{1}{8xy}$

$$m = \iint_{D^*} 8xy \cdot \left|-\frac{1}{8xy}\right| \cdot du\,dv = \iint_{D^*} du\,dv = \text{Area}(D^*) =$$

$$= (25-16)\cdot(9-1) = 9\cdot 8 = 72$$