16.1: Vector Fields

A vector function

$$\vec{r}(t) = \mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

is an example of a function whose domain is a set of real numbers and whose range is a set of vectors in $\mathbb{R}^3$:

$$\mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^3.$$

Consider a type of functions (**vector fields**) whose domain is $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$):

$\mathbb{R} \to \mathbb{R}^2$ $\mathbb{R}^2 \to \mathbb{R}^2$ $\mathbb{R}^3 \to \mathbb{R}^3$

$\mathbb{R} \to \mathbb{R}^3$ $\mathbb{R}^2 \to \mathbb{R}^3$

Vector field over $\mathbb{R}^2$.

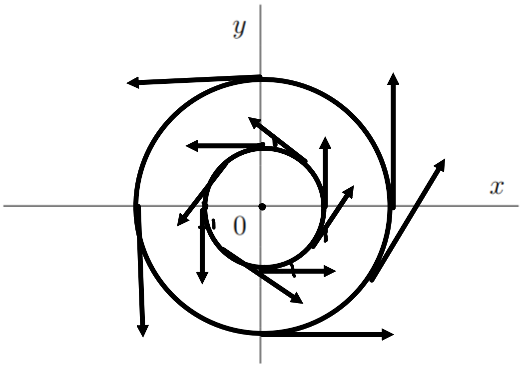
$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

Vector field over $\mathbb{R}^3$:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

A vector field in the plane (for instance), can be visualized as a collection of arrows with a given magnitude and direction, each attached to a point in the plane.

EXAMPLE 1. *Describe the vector field* $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ *by sketching.*



$\vec{r} = (x, y)$	$\vec{F}(x, y)$
$(0,0)$	$\langle 0, 0\rangle$
$(1,0)$	$\langle 0, 1\rangle$
$(0,1)$	$\langle -1, 0\rangle$
$(-1,0)$	$\langle 0, -1\rangle$
$(0,-1)$	$\langle 1, 0\rangle$

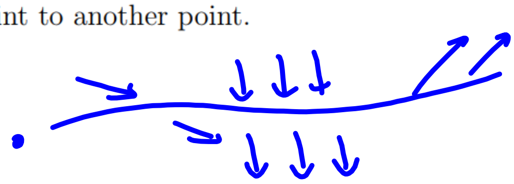
$\vec{r} = \langle x, y\rangle$

$|\vec{F}| = |\langle -y, x\rangle| = \sqrt{y^2 + x^2} = |\vec{r}|$

$\vec{F} \cdot \vec{r} = \langle -y, x\rangle \cdot \langle x, y\rangle = -yx + xy = 0$

$\vec{F} \perp \vec{r}$

Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from one point to another point.

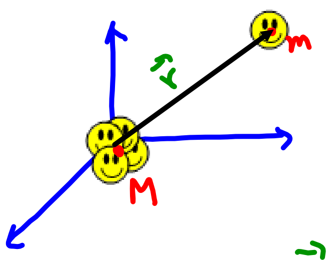


EXAMPLE 2. **Gravitational Field:**

By Newton's Law of Gravitation the magnitude of the gravitational force between two objects with masses m and M is The gravitational force acting on the object at (x, y, z) is

$$|\mathbf{F}| = G\frac{mM}{r^2},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the objects and G is the gravitational constant.



$\vec{r} = \langle x, y, z \rangle$

$\vec{F} = |\vec{F}| \cdot \hat{r} = -|\vec{F}| \dfrac{\vec{r}}{|\vec{r}|}$

$\vec{F}(x,y,z) = -G\dfrac{mM}{r^2} \cdot \dfrac{\vec{r}}{r} = -G\dfrac{mM}{r^3} \cdot \vec{r}$

$\vec{F}(x,y,z) = \left\langle -\dfrac{GmM}{(\sqrt{x^2+y^2+z^2})^3} x, -\dfrac{GmM}{(\sqrt{x^2+y^2+z^2})^3} \cdot y, -\dfrac{GmM}{(\sqrt{x^2+y^2+z^2})^3} \cdot z \right\rangle$

Function $u = f(x, y, z)$ is also called a **scalar field**. Its gradient is also called **gradient vector field**:

$$\mathbf{F}(x, y, z) = \nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$$

EXAMPLE 3. *Find the gradient vector field of* $f(x, y, z) = xyz$.

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle yz, xz, xy \rangle.$$

DEFINITION 4. *A vector field* $\mathbf{F}$ *is called a* **conservative vector field** *if it is the gradient of some scalar function* f *s.t* $\mathbf{F} = \nabla f$. *In this situation* f *is called a* **potential function** *for* $\mathbf{F}$.

$$\vec{F} = \langle P, Q, R \rangle = \langle f_x, f_y, f_z \rangle = \nabla f$$

For instance, the vector field $\mathbf{F}(x, y) = \langle y, x \rangle$ is a conservative vector field with a potential function $f(x, y) = xy$ because

$$f_x = y \qquad \nabla f = \langle y, x \rangle = F(x, y)$$

$$f_y = x$$

REMARK 5. Not all vector fields are conservative, but such fields do arise frequently in Physics.

EXAMPLE 6. *(see Example 2)Let*

$$f(x, y, z) = \frac{GmM}{r\,(x,y,z)},$$

where $r = \sqrt{x^2 + y^2 + z^2}$. *Find its gradient and and answer the questions:*

(a) *Is the gravitational field conservative?*

(b) *What is a potential function of the gravitational field?*

$$\nabla f = \langle f_x, f_y, f_z \rangle = \underbrace{GmM \left\langle -\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3} \right\rangle}_{\text{a gravitational field } \vec{F}(x,y,z)}$$

(a) YES, because there exist a scalar field (function $f(x,y,z)$) such that $\nabla f = \vec{F}$

(b) $f(x,y,z) = \dfrac{GmM}{r}$