16.2: Line Integrals

Line integrals on plane: Let C be a plane curve with parametric equations:

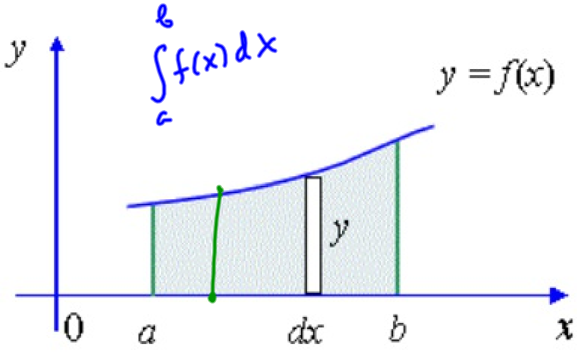
$$x = x(t), y = y(t), \quad \boxed{a \le t \le b,}$$

or we can write the parametrization of the curve as a vector function:

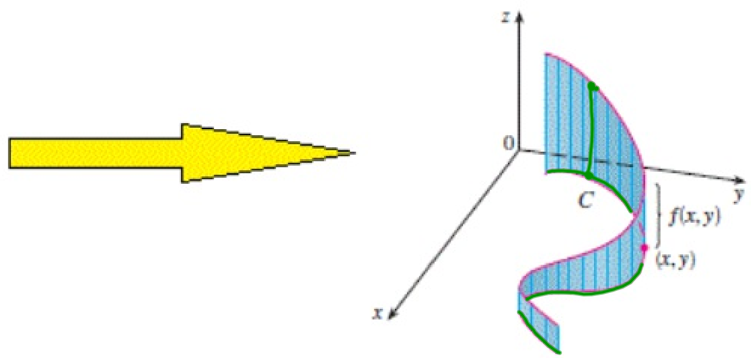
$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \quad \boxed{a \le t \le b.}$$

DEFINITION 1. *The line integral of $f(x, y)$ with respect to arc length, or the* **line integral of f along** C *is*

$$\int_C f(x, y)\, ds \qquad \int_C f(x,y,z)\, ds$$



Definite integral - Area of a flat surface



Line integral - Area of a curved surface

https://brilliant.org/wiki/line-integral/

Recall that the *arc length* of a curve given by parametric equations $x = x(t), y = y(t), \quad a \leq t \leq b$ can be found as

$$L = \int_a^b \mathrm{d}s, \; = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

where

$$\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\, \mathrm{d}t.$$

The line integral is then

$$\int_C f(x,y)\, \mathrm{d}s = \int_a^b f(x(t), y(t)) \sqrt{\;\cdots\;}\; dt$$

If we use the vector form of the parametrization we can simplify the notation up noticing that

$$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

and then

$$\boxed{\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\, \mathrm{d}t = |\vec{r}\,'(t)|\, dt}$$

Using this notation the line integral becomes,

$$\int_C f(x,y)\, \mathrm{d}s = \int_a^b f(x(t), y(t)) |\mathbf{r}'(t)|\, \mathrm{d}t.$$

REMARK 2. The value of the line integral does not depend on the parametrization of the curve, provided that *the curve is traversed exactly once as t increases from a to b.*

Let us emphasize that $\quad \mathrm{d}s = |r'(t)|\,\mathrm{d}t = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,\mathrm{d}t.$

EXAMPLE 3. *Evaluate the line integral* $\int_C y\,\mathrm{d}s$, *where* $C : x = t^3, y = t^2, 0 \le t \le 1.$

$x' = 3t^2 \quad , \quad y' = 2t$

$ds = \sqrt{(3t^2)^2 + (2t)^2}\; dt$

$ds = \underbrace{\sqrt{9t^4 + 4t^2}}_{t\sqrt{9t^2+4}}\; dt \qquad (t \geq 0)$

$$\int_C y\,ds = \int_0^1 \underbrace{t^2}_{\frac{u-4}{9}}\, t\sqrt{9t^2+4}\; dt = \int_4^{13} \frac{u-4}{9}\sqrt{u}\,\frac{du}{18}$$

$u = 9t^2 + 4$

$du = 18t\, dt$

$4 \le u \le 13$

$$= \int_4^{13} \frac{u\sqrt{u} - 4\sqrt{u}}{9 \cdot 18}\, du = \ldots$$

Line integrals in space: Let C be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \leq t \leq b,$$

or

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b.$$

The **line integral of f along C** is

$$\int_C f(x, y, z)\, \mathrm{d}s = \int_a^b f(x(t), y(t), z(t))|r'(t)|\, \mathrm{d}t.$$

Here

$$\boxed{\mathrm{d}s = |r'(t)|\, \mathrm{d}t} = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, \mathrm{d}t.$$

EXAMPLE 4. *Evaluate the line integral* $\int_C (x+y+z)\,\mathrm{d}s$, *where* C *is the line segment joining the points* $A(-1,1,2)$ *and* $B(2,3,1)$.

$\vec{V} = \overrightarrow{AB} = \langle 2-(-1), 3-1, 1-2\rangle = \langle 3, 2, -1\rangle$

$$C \begin{cases} x = -1 + 3t \\ y = 1 + 2t \\ z = 2 - t \\ 0 \le t \le 1 \end{cases} \Rightarrow \begin{aligned} x' &= 3 \\ y' &= 2 \\ z' &= -1 \end{aligned} \Rightarrow ds = \sqrt{3^2 + 2^2 + (-1)^2}\,dt = \sqrt{14}\,dt$$

$$\int_C (x+y+z)\,dS = \int_0^1 \Big(\big(-1+3t\big) + (1+2t) + (2-t)\Big)\sqrt{14}\,dt =$$

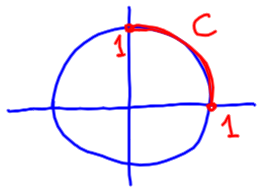
$$= \sqrt{14}\int_{-1}^{0} (6+4t)\,dt = \ldots$$

Physical interpretation of a line integral: Let $\rho(x,y,z)$ represents the linear density at a point (x,y,z) of a thin wire shaped like a curve C. Then the **mass** m of the wire is:

$$\boxed{m = \int_C \rho(x,y,z)\,\mathrm{d}s.}$$

counterclockwise direction.

EXAMPLE 5. *A thin wire with the linear density* $\rho(x,y) = x^2 + 2y^2$ *takes the shape of the curve* C *which consists of the arc of the circle* $x^2+y^2=1$ *from* $(1,0)$ *to* $(0,1)$. *Find the mass of the wire.*



$C: x = t, \quad y = \sqrt{1-t^2}, \quad 0 \le t \le 1$ example of parameterizati

We will use another parameterization:

$x = \sin t, \quad y = \cos t, \quad 0 \le t \le \pi/2$

$\vec{r}(t) = \langle \sin t, \cos t \rangle \Rightarrow \vec{r}\,'(t) = \langle \cos t, -\sin t \rangle$

$|\vec{r}\,'(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$$m = \int_C \rho(x,y)\,ds = \int_C (x^2+2y^2)\,ds =$$

$$= \int_0^{\pi/2} (\sin^2 t + 2\cos^2 t)\,\underbrace{|\vec{r}\,'(t)|}_{=1}\,dt$$

$$= \int_0^{\pi/2} (\underbrace{\sin^2 t + \cos^2 t}_{1} + \cos^2 t)\,dt$$

$$= \frac{\pi}{2} + \int_0^{\pi/2} \underbrace{\cos^2 t}_{\frac{1+\cos 2t}{2}}\,dt = \ldots$$

Line integrals with respect to x, y, and z. Let C be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \le t \le b,$$

$dx = x'(t)\,dt$

The **line integral of f with respect to** x is,

$$\int_C f(x,y,z)\,\mathrm{d}x = \int_a^b f(x(t),y(t),z(t))\underbrace{x'(t)\,\mathrm{d}t}_{dx}.$$

The **line integral of f with respect to** y is,

$$\int_C f(x,y,z)\,\mathrm{d}y = \int_a^b f(x(t),y(t),z(t))\underbrace{y'(t)\,\mathrm{d}t}_{dy}.$$

The **line integral of f with respect to** z is,

$$\int_C f(x,y,z)\,\mathrm{d}z = \int_a^b f(x(t), y(t), z(t))\,\underbrace{z'(t)\,dt}_{dz}$$

These two integral often appear together by the following notation:

$$\boxed{\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z}$$

or

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

where $\vec{F}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

EXAMPLE 6. *Compute*

$$\int_C \vec{F}\cdot d\vec{r} = I = \int_C \underbrace{-\frac{y}{x^2+y^2}}_{P(x,y)}\,\mathrm{d}x + \underbrace{\frac{x}{x^2+y^2}}_{Q(x,y)}\,\mathrm{d}y = \int_C -y\,dx + x\,dy$$

where C is the circle $x^2+y^2=1$ oriented in the counterclockwise direction.

$C: \; x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$

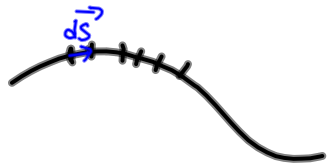
$dx = x'\,dt = -\sin t\,dt$

$dy = y'\,dt = \cos t\,dt$

$$I = \int_0^{2\pi} -\sin t \underbrace{(-\sin t)\,dt}_{dx} + \cos t \underbrace{\cos t\,dt}_{dy} = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= \int_0^{2\pi} dt = \boxed{2\pi}$$

Line integrals of vector fields.



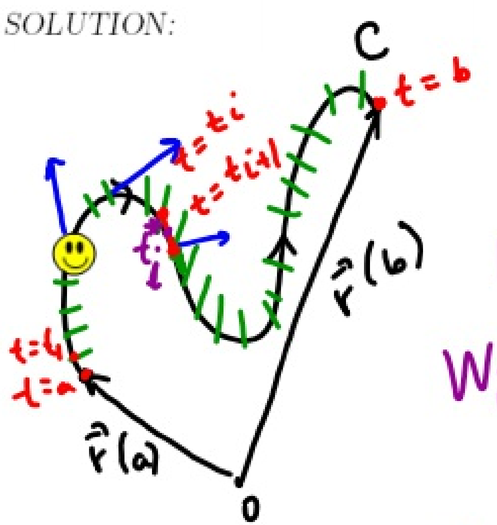
PROBLEM: Given a continuous force field,

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k},$$

such as a gravitational field. Find the work done by the force $\mathbf{F}$ in moving a particle along a curve

$$C: \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad a \le t \le b.$$

SOLUTION:



Recall

$[t_i, t_{i+1}]$

$$W = \vec{F}\cdot\vec{D}$$

$$W_i \approx \vec{F}(\vec{r}(t_i^*))\cdot\Delta\vec{r}(t_i)$$

where $\Delta\vec{r}(t_i) \approx \vec{r}(t_{i+1}) - \vec{r}(t_i)$

$$\|P\| = \max_i |\Delta\vec{r}(t_i)|$$

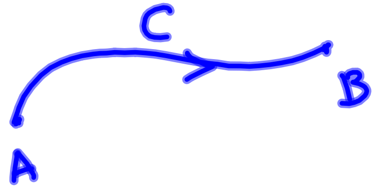
$$W = \lim_{\|P\|\to 0} \sum_i W_i$$

$$W = \lim_{\|P\|\to 0} \sum_i \vec{F}(\vec{r}(t_i^*))\cdot\Delta\vec{r}(t_i) = \int_C \vec{F}\cdot d\vec{r}$$

DEFINITION 7. *Let* $\mathbf{F}$ *be a continuous vector field defined on a curve* C *given by a vector function* $\mathbf{r}(t)$, $a \leq t \leq b$. *Then the* **line integral of F along** C *is*

dot product

$$\int_C \vec{F}(x,y,z) \cdot d\vec{r} = \int_C \vec{\mathbf{F}} \cdot \, \mathrm{d}\mathbf{r}(t) = \int_a^b \underbrace{\mathbf{F}(\mathbf{r}(t))}_{\text{vector}} \cdot \underbrace{\mathbf{r}'(t)}_{\text{vector}} \, \mathrm{d}t.$$



REMARK 8. Note that this integral depends on the curve orientation:

$$\int_{-C} \mathbf{F} \cdot \, \mathrm{d}\mathbf{r}(t) = -\int_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{r}(t)$$

$\vec{F} = \langle P, Q, R \rangle$

Relationship between line integrals of vector fields and line integrals with respect to x, y, and z.

$$\underbrace{\int_C \mathbf{F} \cdot d\mathbf{r}(t)}_{\text{2-nd type line integral}} = \int_C \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle =$$

$$= \int_C P\,dx + Q\,dy + R\,dz$$

EXAMPLE 9. *Find the work done by the force field* $\mathbf{F}(x,y,z) = \langle xy, yz, xz \rangle$ *in moving a particle along the curve* $C : \mathbf{r}(t) = \langle \underbrace{t}_{x}, \underbrace{t^2}_{y}, \underbrace{t^3}_{z} \rangle$, $\underline{0 \le t \le 1}$.

$$W = \int_C \vec{F}(x,y,z) \cdot d\vec{r} = \int_C \langle xy, yz, xz \rangle \cdot \langle dx, dy, dz \rangle =$$

$dx = t' \, dt = dt$

$$= \int_C xy\,dx + yz\,dy + xz\,dz =$$

$$= \int_0^1 \left(t \cdot t^2 \cdot \quad + t^2 \cdot t^3 \cdot 2t \quad + t \cdot t^3 \cdot 3t^2 \right) dt =$$

$$= \int_0^1 \left(t^3 + 2t^6 + 3t^6 \right) dt = \frac{1}{4} + \frac{5}{7} = \boxed{\frac{27}{28}}$$