16.3: The fundamental Theorem for Line Integrals

16.4: Green's Theorem

see sec. 16.1

DEFINITION 1. *A vector field* **F** *is called a* **conservative vector field** *if it is the gradient of some scalar function f s.t* $\mathbf{F} = \nabla f$. *In this situation f is called a* **potential function** *for* **F**.

Recall Part 2 of the Fundamental Theorem of Calculus:

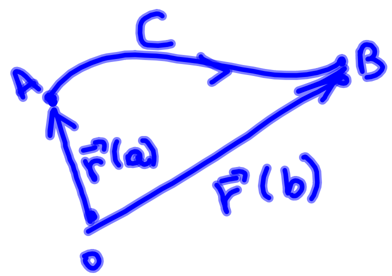
$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a),$$

where F' is continuous on $[a, b]$.

- **The fundamental Theorem for Line Integrals:** *Let C be a smooth curve given by $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables and ∇f is continuous on C. Then*

$$\int_A^B \nabla f \cdot d\vec{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

$$= f(B) - f(A)$$

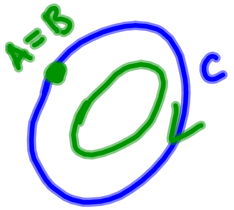


- **The fundamental Theorem for Line Integrals:** *Let C be a smooth curve given by $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables and ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). = f(B) - f(A)$$

$\vec{F}$ is conservative $\Rightarrow$ $\vec{F} = \nabla f$ for some f $\Rightarrow$

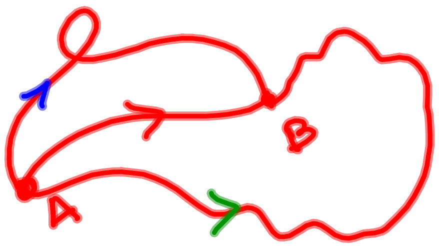
REMARK 2. If C is a closed curve then



$$\oint_C \nabla f \cdot d\vec{r} = 0$$

$\Rightarrow$ $\oint_C \vec{F} \cdot d\vec{r} = \oint_C \nabla f \cdot d\vec{r} = 0$ for every closed curve C.

COROLLARY 3. *If F is a conservative vector field and C is a curve with initial point A and terminal point B then:*



the integral $\int_C \nabla f \cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r}$ does not depend on path of integration.

sec. 16.1

EXAMPLE 6. *(see Example 2) Let*

$$f(x,y,z) = \frac{GmM}{r(x,y,z)},$$

where $r = \sqrt{x^2+y^2+z^2}$. *Find its gradient and and answer the questions:*

(a) *Is the gravitational field conservative?*

(b) *What is a potential function of the gravitational field?*

$$\nabla f = \langle f_x, f_y, f_z \rangle = \underbrace{GmM \left\langle -\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3} \right\rangle}_{\text{a gravitational field } \vec{F}(x,y,z)}$$

(a) YES, because there exist a scalar field (function $f(x,y,z)$) such that $\nabla f = \vec{F}$

(b) $f(x,y,z) = \frac{GmM}{r}$

EXAMPLE 4. *Find the work done by the gravitational field*

$$\mathbf{F}(x,y,z) = -\frac{GmM}{(x^2+y^2+z^2)^{3/2}} \langle x,y,z \rangle$$

in moving a particle with mass m from the point $\underbrace{(1,2,2)}_{A}$ *to the point* $\underbrace{(3,4,12)}_{B}$ *along a piecewise-smooth curve C.*

$$W = \int_C \vec{F} \cdot d\vec{r}$$

By Ex.6 (Sec. 16.1), $\vec{F}$ is conservative and $f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}}$ is its potential function, i.e. $\boxed{\nabla f = \vec{F}.}$

$$W = \int_C \nabla f \cdot dr = \int_{(1,2,2)}^{(3,4,12)} \nabla f \cdot d\vec{r} \overset{\text{FTLI}}{=} f(3,4,12) - f(1,2,2) =$$

$$= \frac{GmM}{\sqrt{3^2+4^2+12^2}} - \frac{GmM}{\sqrt{1^2+2^2+2^2}} = \ldots$$

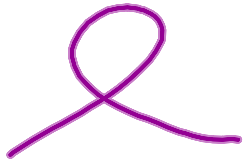
Notations And Definitions:

DEFINITION 5. *A piecewise-smooth curve is called a* **path**.

- **Types of curves:**

simple not closed

not simple not closed



simple closed

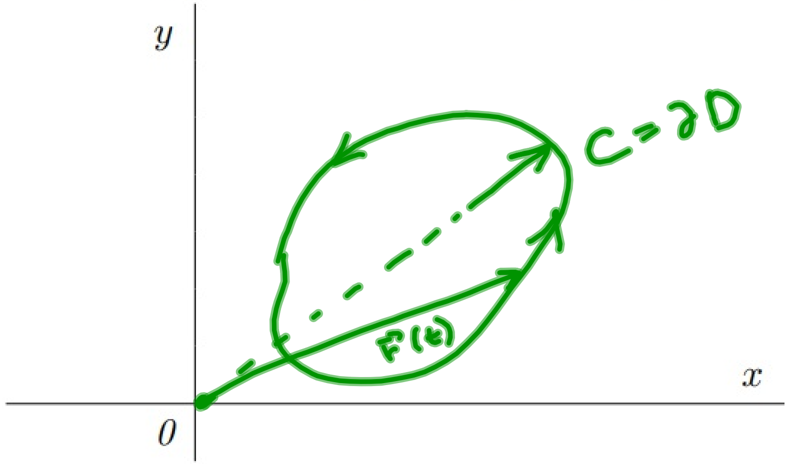
not simple, closed

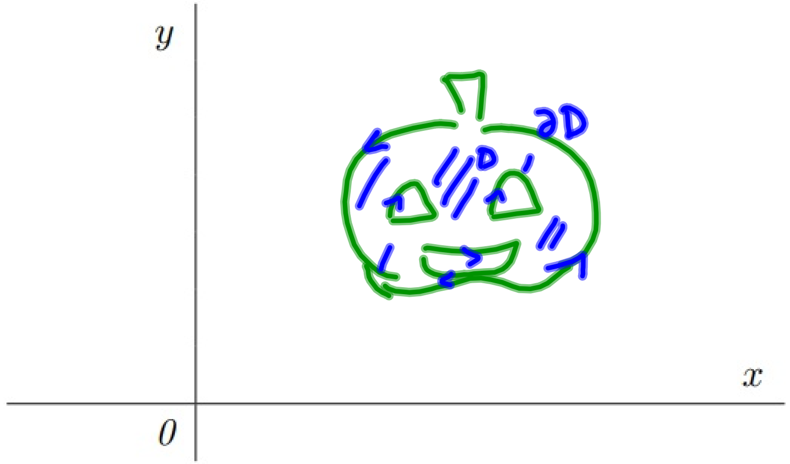
- **Types of regions:**

simply connected

not simply connected

- **Convention:** *The* **positive orientation** *of a simple closed curve C refers to a single counter-clockwise traversal of C. If C is given by $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}, a \leq t \leq b$, then the region D bounded by C is always on the left as the point $\mathbf{r}(t)$ traverses C.*





- *The positively oriented boundary curve of D is denoted by ∂D.*

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ vector field

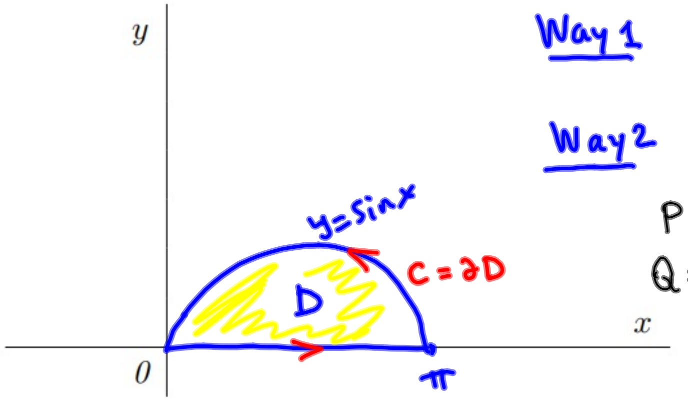
•**GREEN's THEOREM***:* Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region that contains D, then

$$\oint_{\partial D} \vec{F}\cdot d\vec{r} = \underbrace{\oint_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y}_{\text{line integral}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A}_{\text{double integral}}.$$

EXAMPLE 6. *Evaluate:*

$$I = \oint_C e^x(1-\cos y)\,\mathrm{d}x - e^x(1-\sin y)\,\mathrm{d}y$$

where C is the boundary of the domain $D = \{(x,y) : 0 \le x \le \pi, 0 \le y \le \sin x\}$.



Way 1 Parameterize C ☹

Way 2 Use Green's Th. ☺

$P = e^x(1-\cos y) = e^x - e^x \cos y$

$Q = -e^x(1-\sin y)$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \underbrace{-e^x(1-\sin y)} - \underbrace{e^x \sin y} =$$

$$= -e^x + e^x \sin y - e^x \sin y = -e^x$$

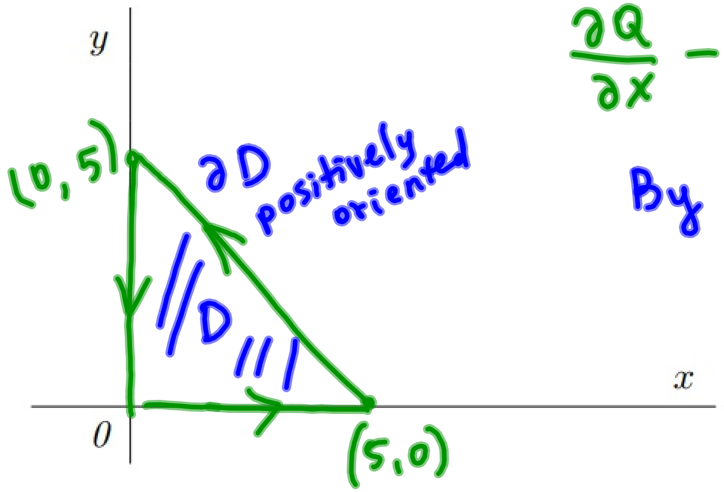
By Green's Theorem

$$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D -e^x \, dA$$

$$= -\int_0^\pi \int_0^{\sin x} e^x \, dy\, dx = -\int_0^\pi e^x \sin x \, dx = \dots$$

EXAMPLE 7. *Let C be a triangular curve consisting of the line segments from $(0,0)$ to $(5,0)$, from $(5,0)$ to $(0,5)$, and from $(0,5)$ to $(0,0)$. Evaluate the following integral:*

$$I_1 = \oint_C \underbrace{(x^2y + \frac{1}{2}y^2)}_{P}\, \mathrm{d}x + \underbrace{(xy + \frac{1}{3}x^3 + 3x)}_{Q}\, \mathrm{d}y$$



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y + x^2 + 3 - (x^2 + y) = 3$$

By Green's theorem

$$I_1 = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 3\, dA$$

$$= 3 \iint_D dA = 3 \cdot (\text{Area}(D))$$

$$= 3 \cdot \frac{5 \cdot 5}{2} = \frac{75}{2}$$

SUMMARY: Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a vector field on an open simply connected domain D. Suppose that P and Q have continuous partial derivatives through D. Then the facts below are equivalent.

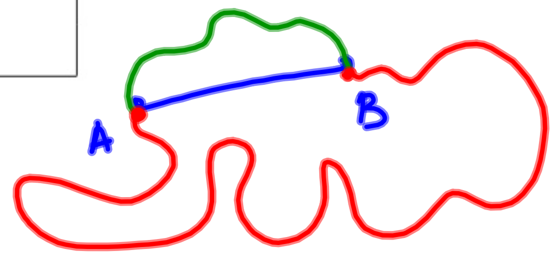
The field $\mathbf{F}$ is **conservative**on D $\iff$ There exists f s.t. $\nabla f = \mathbf{F}$

potential function

$\Updownarrow$

The field $\mathbf{F}$ is **conservative** on D $\iff$ $\int_{A\breve{B}} \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D

$= f(B) - f(A)$



$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$

The field $\mathbf{F}$ is **conservative** on D $\iff$ $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ throughout D

The field $\mathbf{F}$ is **conservative** on D $\iff$ $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve C in D

$\|$ Green's theorem

$$\iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{=0} dA = 0$$

EXAMPLE 8. *Determine whether or not the vector field is conservative:*

(a) $\mathbf{F}(x,y) = \langle \underbrace{x^2+y^2}_{P}, \underbrace{2xy}_{Q} \rangle$.

$$\frac{\partial Q}{\partial x} = 2y$$
$$\parallel$$
$$\frac{\partial P}{\partial y} = 2y$$

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ for $\underline{\underline{\text{all}}}$ x, y.

$\vec{F}$ is conservative.

(b) $\mathbf{F}(x,y) = \langle \underbrace{x^2+3y^2+2}_{P}, \underbrace{3x+ye^y}_{Q} \rangle$

$$\frac{\partial P}{\partial y} = 6y \neq \frac{\partial Q}{\partial x} = 3$$

$\vec{F}$ is not conservative.

EXAMPLE 9. *Given* $\mathbf{F}(x,y) = \sin y\mathbf{i} + (x\cos y + \sin y)\mathbf{j}$.

(a) *Show that* $\mathbf{F}$ *is conservative.*

$P = \sin y$

$Q = x\cos y + \sin y$

$$\frac{\partial P}{\partial y} = \cos y = \frac{\partial Q}{\partial x} \quad \text{for all } x, y.$$

Find a potential of $\vec{F}$.

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$

Find $f(x,y)$ such that

$$\begin{cases} f_x = \sin y & (1) \\ f_y = x\cos y + \sin y & (2) \end{cases}$$

$\langle f_x, f_y \rangle = \langle P, Q \rangle$

To find f solve $f_x = P$, $f_y = Q$

$$(1) \Rightarrow f(x,y) = \int \sin y \, dx = \underbrace{x\sin y + C_1(y)}_{\text{Plug in (2)}} \quad (3)$$

$$\underbrace{\frac{\partial}{\partial y}\left(x\sin y + C_1(y)\right)}_{f_y} \overset{(2)}{=} \underbrace{x\cos y + \sin y = f_y}$$

$$\cancel{x\cos y} + C_1'(y) = \cancel{x\cos y} + \sin y$$

$$C_1(y) = \int \sin y \, dy + C$$

$$C_1(y) = -\cos y + C \quad (4)$$

$(4) \to (3)$:

$$\boxed{f(x,y) = \underline{x\sin y - \cos y + C}}$$

Answer: for example:

$$\underbrace{f(x,y) = x\sin y - \cos y}_{\text{a potential of } \vec{F}(x,y)}.$$

(c) *Find the work done by the force field* $\mathbf{F}$ *in moving a particle from the point* $(3,0)$ *to the point* $(0,\pi/2)$.

$$W = \int_{(3,0)}^{(0,\pi/2)} \vec{F}\cdot d\vec{r} \underset{\vec{F}\text{ is conservative}}{\overset{(b)}{=}} \int_{(3,0)}^{(0,\pi/2)} \nabla f\cdot d\vec{r} \overset{\text{FTLI}}{=} f(0,\tfrac{\pi}{2}) - f(3,0) \overset{(b)}{=}$$

$$= 0\sin\frac{\pi}{2} - \cos\frac{\pi}{2} - (3\sin 0 - \cos 0) = 1$$

(d) *Evaluate* $\oint_C \mathbf{F}\,d\mathbf{r}$ *where* C *is an arbitrary path in* $\mathbb{R}^2$.

$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

$$\left(\oint_C \vec{F}\cdot d\vec{r} = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\overset{(a)}{=}\,0} dA = 0 \right)$$

EXAMPLE 10. *Given*

$$\vec{F}(x,y,z) = \mathbf{F} = \langle \overbrace{2xy^3 + z^2}^{P}, \overbrace{3x^2y^2 + 2yz}^{Q}, \overbrace{y^2 + 2xz}^{R} \rangle .$$

Find a $\underbrace{\textit{function}}_{\text{potential}}$ f *s.t.* $\nabla f = \mathbf{F}$ (In other words, show that $\vec{F}$ is conservative)

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle P, Q, R \rangle$$

$$\begin{cases} f_x = 2xy^3 + z^2 & (1) \\ f_y = 3x^2y^2 + 2yz & (2) \\ f_z = y^2 + 2xz & (3) \end{cases}$$

$(1) \Rightarrow f(x,y,z) = \int (2xy^3 + z^2)\,dx + C_1(y,z) =$

$= x^2y^3 + xz^2 + C_1(y,z)$

$(2) \Rightarrow$

$$\frac{\partial}{\partial y}\left(x^2y^3 + xz^2 + C_1(y,z)\right) = 3x^2y^2 + 2yz$$

$$\cancel{3x^2y^2} + \frac{\partial C_1(y,z)}{\partial y} = \cancel{3x^2y^2} + 2yz$$

$$C_1(y,z) = \int 2yz\,dy + C_2(z)$$

$$C_1(y,z) = y^2z + C_2(z)$$

So,

$$f(x,y,z) = x^2y^3 + xz^2 + y^2z + C_2(z) \quad (4)$$

$(4) \to (3)$:

$$\frac{\partial}{\partial z}\left(x^2y^3 + xz^2 + y^2z + C_2(z)\right) = y^2 + 2xz$$

$$\cancel{2xz} + \cancel{y^2} + \frac{\partial C_2(z)}{\partial z} = \cancel{y^2} + \cancel{2xz}$$

$$\boxed{C_2(z) = C}$$

$$\boxed{f(x,y,z) = x^2y^3 + xz^2 + y^2z} + C$$