14.5: Curl and Divergence

Introduce the vector differential operator ∇ as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field on R^3 and the partial derivatives of P, Q, R all exist, then the **curl** of $\mathbf{F}$ is the *vector field* on R^3 defined by

$$\text{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}\hat{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}\hat{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}\hat{k}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j} + \underline{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}\hat{k}$$

EXAMPLE 1. *Find the curl of the vector field*

$$\mathbf{F}(x,y,z) = \langle \underbrace{xy}_{P}, \underbrace{x^2}_{Q}, \underbrace{yz}_{R} \rangle.$$

$$\text{curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2 & yz \end{vmatrix} = \hat{i}\left(\frac{\partial(yz)}{\partial y} - \frac{\partial(x^2)}{\partial z}\right) -$$

$$-\hat{j}\left(\frac{\partial}{\partial x}(yz) - \frac{\partial(xy)}{\partial z}\right) +$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy)\right) =$$

$$= \hat{i}(z-0) - \hat{j}(0-0) + \hat{k}(2x - x)$$

$$= \langle z, 0, x \rangle.$$

Question *What is the curl of a two-dimensional vector field*

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} \ ?$$

Answer:

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \hat{i}\left(\frac{\partial \cdot 0}{\partial y} - \underbrace{\frac{\partial Q(x,y)}{\partial z}}_{=0}\right) -$$

$$-\hat{j}\left(\frac{\partial 0}{\partial x} - \underbrace{\frac{\partial P(x,y)}{\partial z}}_{=0}\right) +$$

$$+\hat{k}\left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right)$$

$$\left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

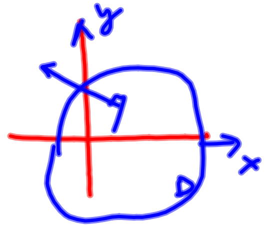
$$1 = \hat{k}\cdot\hat{k}$$
$$\langle 0,0,1\rangle \cdot \langle 0,0,1\rangle = 1.$$

CONCLUSION: Green's Theorem in vector form:

$$\oint_{\partial D} P\,dx + Q\,dy = \oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cdot 1\, dA$$

$$= \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}}_{\text{curl}\,\vec{F}} \cdot \hat{k}\, dA$$

$$\boxed{\oint_{\partial D} \vec{F}\cdot d\vec{r} = \iint_D \text{curl}\,\vec{F} \cdot \underbrace{\hat{k}}_{\substack{\text{normal}\\ \text{to the plane}\\ \text{containing } D}}\, dA}$$



THEOREM 2. *If a function $f(x, y, z)$ has continuous partial derivatives of second order then*

$$\operatorname{curl}(\nabla f) = \vec{0}.$$

Proof. <u>Way 1</u> $\operatorname{curl}(\nabla f) = \operatorname{curl}\langle f_x, f_y, f_z\rangle = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \ldots = \vec{0}$

<u>Way 2</u> $\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) = \underbrace{(\nabla \times \nabla)}_{= \vec{0}} f = \vec{0}$

COROLLARY 3. *If* $\mathbf{F}$ *is conservative, then* $\operatorname{curl}\mathbf{F} = 0$.

The proof of the Theorem below requires Stokes' Theorem (Section 14.8).

THEOREM 4. *If* $\mathbf{F}$ *is a vector field defined on* R^3 *whose component functions have continuous partial derivatives and* $\mathrm{curl}\mathbf{F} = 0$*, then* $\mathbf{F}$ *is a conservative vector field.*

EXAMPLE 5. *Let* $\mathbf{F}(x,y,z) = \langle x^9, y^9, z^9 \rangle$.

(a) *Show that* $\mathbf{F}$ *is conservative.*

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^9 & y^9 & z^9 \end{vmatrix} = \hat{\imath}\left(\underbrace{\frac{\partial}{\partial y}(z^9)}_{=0} - \underbrace{\frac{\partial}{\partial z}(y^9)}_{=0}\right) - \hat{\jmath}(0-0) + \hat{k}(0-0) = \vec{0}$$

So, curl $\vec{F} = 0 \Rightarrow \vec{F}$ is conservative.

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$.

<u>Way 1</u> solve system of diff. equations

<u>Way 2</u> Guess $f(x,y,z) = \frac{x^{10}}{10} + \frac{y^{10}}{10} + \frac{z^{10}}{10}$

Then $f_x = x^9$, $f_y = y^9$, $f_z = z^9$

(c) *Evaluate* $\int_{(1,0,1)}^{(-1,-1,-1)} \vec{\mathbf{F}} \cdot d\mathbf{r} \overset{(b)}{=} \int_{(1,0,1)}^{(-1,-1,-1)} \nabla f \cdot d\vec{r} \overset{\text{FTLI}}{=} f(-1,-1,-1) - f(1,0,1)$

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} - \frac{1}{10} - \frac{0}{9} - \frac{1}{10} = \frac{1}{10}$$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field on R^3 and the partial derivatives P_x, Q_y, R_z exist, then the **divergence of F** is the *scalar field* on defined by

dot product

$$\mathrm{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle .$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

EXAMPLE 6. *Find the divergence of the vector field*

$$\mathbf{F}(x,y,z) = \langle \sin(xyz), x^2, yz \rangle .$$

$$\mathrm{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(\sin(xyz) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(yz)$$

$$= \underbrace{yz\cos(xyz) + y}_{\text{scalar field}}$$

THEOREM 7. *If the components of a vector field* $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ *has continuous partial derivatives of second order then*

$$\text{div curl } \mathbf{F} = 0.$$

Proof. $\text{div}(\text{curl } \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$

$\vec{a} \cdot \underbrace{(\vec{a} \times \vec{b})}_{\vec{c}} = 0$

$\vec{c} \perp \vec{a}$ and $\vec{c} \perp \vec{b}$

$\Downarrow$

$\vec{a} \cdot \vec{c} = 0$

EXAMPLE 8. *Is there a vector field* $\mathbf{G}$ *on* R^3 *s.t.* $\boxed{\text{curl } G = \langle yz, xyz, zy \rangle}$*?*

If such a field $\vec{G}$ exists, then by theorem 7,

$\text{div}(\text{curl } \vec{G}) = 0.$

However, $\text{div}(\text{curl } \vec{G}) = \text{div}\langle yz, xyz, zy \rangle$

$= \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(zy)$

$= 0 + xz + y = xz + y \neq 0$

for all x, y, z.

a contradiction.

Such field $\vec{G}$ does not exist.

Th. 8

$\text{div}(\text{curl } \vec{F}(x,y,z)) = 0$

for all $x, y, z \in \mathbb{R}^3$