16.6: Parametric surfaces and their areas

Consider a continuous vector valued function of two variables

surface in vector form

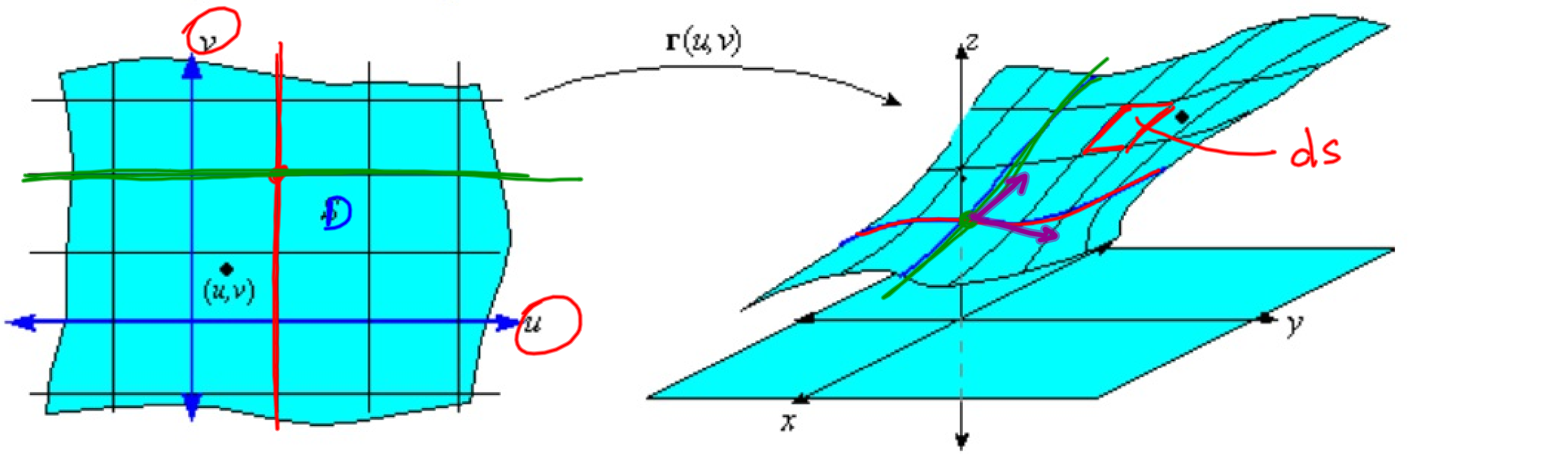
$$\vec{\mathbf{r}}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D.$$

parameters domain

Parametric surface:

$$S: \ x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \quad (u,v) \in D.$$

In other words, the surface S is traces out by the position vector $\mathbf{r}(u,v)$ as (u,v) moves throughout the region D.



EXAMPLE 1. *Determine the surface given by the parametric representation*

parameters domain

$$\mathbf{r}(u,v) = \langle u, u\cos v, u\sin v\rangle\,, \quad \boxed{1 \le u \le 5, \quad 0 \le v \le 2\pi}$$

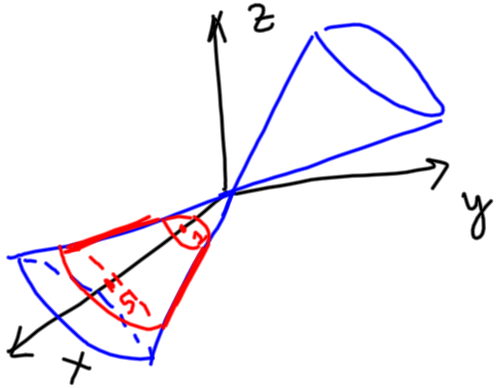
$x = u$

$y = u\cos v$

$z = u\sin v$

$\Rightarrow y^2 + z^2 = u^2\cos^2 v + u^2\sin^2 v$

$y^2 + z^2 = \underbrace{u^2}_{x} (\underbrace{\cos^2 v + \sin^2 v}_{=1})$



$\boxed{y^2 + z^2 = x^2}$ cone

$1 \le x \le 5$

So, the given surface is a portion of the cone $y^2+z^2=x^2$ between the planes $x=1$ and $x=5$

EXAMPLE 2. *Give parametric or vector representations for each of the following surfaces:*

(a) *the cylinder:* $x^2 + y^2 = 9$, $\boxed{1 \le z \le 5.}$

Use cylindrical coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x^2 + y^2 = 9$

$r^2 = 9$

$r = 3$

parametric

$$\boxed{\begin{aligned} x &= 3\cos\theta \\ y &= 3\sin\theta \\ z &= z \\ 1 \le z \le 5, &\quad 0 \le \theta \le 2\pi \end{aligned}}$$

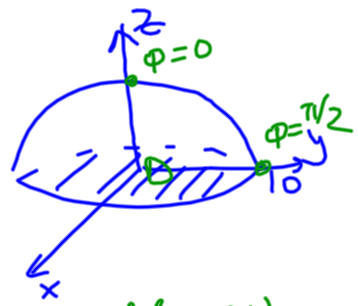
OR

vector representation

$\vec{r}(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z \rangle$, $(\theta, z) \in D$,

where $D = \{ (\theta, z) \mid 0 \le \theta \le 2\pi,\ 1 \le z \le 5 \}$

(b) *the upper half-sphere:* $z = \sqrt{100 - x^2 - y^2}$.



Way 1 $\vec{r}(x,y) = \langle x, y, \sqrt{100 - x^2 - y^2} \rangle$

$(x,y) \in D$, where

$D = \{(x,y) \mid x^2+y^2 \le 100\}$ projection of the half-sphere onto the xy-plane.

Way 2 Use spherical coordinates

$x = \rho \sin\varphi \cos\theta$ $\rho = 10$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

$$\boxed{\begin{gathered}\vec{r}(\theta,\varphi) = \langle 10 \sin\varphi\cos\theta, 10\sin\varphi\sin\theta, 10\cos\varphi \rangle \\ (\theta,\varphi) \in D, \text{ where} \\ D = \{(\theta,\varphi) \mid 0 \le \theta \le 2\pi, 0 \le \varphi \le \pi/2\}\end{gathered}}$$

CONCLUSION: To parametrize surface we may use polar, cylindrical or spherical coordinates, or

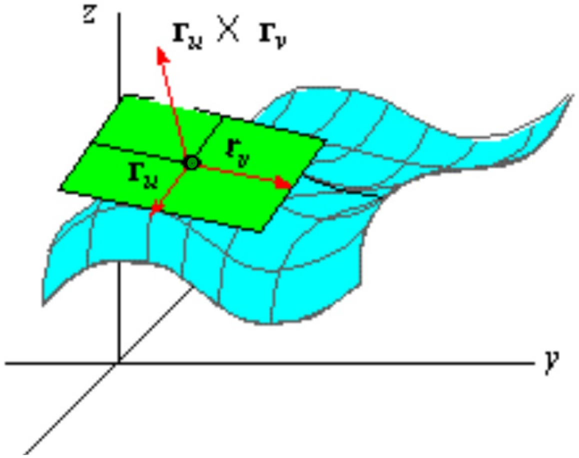
In all cases the parameters domain D is projection onto the

Special cases

- $z = f(x, y) \longrightarrow \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$ xy-plane
- $y = f(x, z) \longrightarrow \mathbf{r}(x, z) = x\mathbf{i} + f(x, z)\mathbf{j} + z\mathbf{k}$ xz-plane
- $x = f(y, z) \longrightarrow \mathbf{r}(y, z) = f(y, z)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ yz-plane

• **Tangent planes**:

PROBLEM: Find a normal vector to the tangent plane to a parametric surface S given by a vector function $\mathbf{r}(u,v)$ at a point P_0 with position vector $\mathbf{r}(u_0,v_0)$, i.e. $P_0(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0))$



The **normal vector**

$$\mathbf{N} = \mathbf{N}(u,v) = \vec{r}_u \times \vec{r}_v$$

If a normal vector is not $\mathbf{0}$ then the surface S is called **smooth** (it has no "corner").

Special Case: a surface S given by a *graph* $\boxed{z = f(x,y)}$. Then one can choose the following parametrization of S:

$$\mathbf{r}(x,y) = \langle x, y, f(x,y) \rangle$$

and the then the normal vector is

$$\mathbf{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$$

EXAMPLE 3. *Find the tangent plane to the surface with parametric equations* $x = uv + 1, y = ue^v, z = ve^u$ *at the point* $(1, 0, 0)$. point on the plane

$\vec{r}(u,v) = \langle uv+1, ue^v, ve^u \rangle$

Find (u,v) such that $\vec{r}(u,v) = \langle 1,0,0 \rangle$

$$\left.\begin{aligned} uv+1 &= 1 \\ ue^v &= 0 \\ ve^v &= 0 \end{aligned}\right\} \Rightarrow (u,v) = (0,0)$$

$\vec{r}_u = \langle v, e^v, ve^u \rangle \Rightarrow \vec{r}_u(0,0) = \langle 0,1,0 \rangle$

$\vec{r}_v = \langle u, ue^v, e^u \rangle \Rightarrow \vec{r}_v(0,0) = \langle 0,0,1 \rangle$

$$\vec{N}(0,0) = \vec{r}_u^{(0,0)} \times \vec{r}_v^{(0,0)} = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 1,0,0 \rangle$$ normal

$1(x-1) + 0(y-0) + 0(z-0) = 0$

$\boxed{x = 1}$

- **Surface Area:**

Consider a smooth surface S given by

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D,$$

then

$$\mathrm{d}S = |N(u,v)|\mathrm{d}u\mathrm{d}v = |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

and the **surface area**

$$A(S) = \iint_S \mathrm{d}S = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, \overset{dudv}{\mathrm{d}A}.$$

REMARK 4. *Special Case*: a surface S given by a *graph* $z = f(x,y)$ we have

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

and

$$\mathrm{d}S = |\mathbf{N}(x,y)|\,\mathrm{d}A = |\langle f_x, f_y, -1\rangle|\, dx\,dy$$

$$= \sqrt{f_x^2 + f_y^2 + 1}\; dx\,dy$$

EXAMPLE 5. *Find the surface area of the surface*

$$S: \quad x = uv, \quad y = u+v, \quad z = u-v, \qquad u^2+v^2 \le 1.$$

$$\mathcal{D} = \{(u,v) \mid u^2+v^2 \le 1\}$$

$$\vec{r}(u,v) = \langle uv,\ u+v,\ u-v \rangle$$

$$\vec{N}(u,v) = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} = \langle -1-1, -(-v-u), v-u \rangle$$

$$= \langle -2,\ v+u,\ v-u \rangle$$

$$|\vec{N}(u,v)| = |\vec{r}_u \times \vec{r}_v| = |\langle -2,\ v+u,\ v-u \rangle|$$

$$= \sqrt{4 + (v+u)^2 + (v-u)^2}$$

$$= \sqrt{4 + v^2 + 2\cancel{uv} + u^2 + v^2 - 2\cancel{uv} + u^2}$$

$$= \sqrt{4 + 2(u^2+v^2)}$$

Surface area

$$A(S) = \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

$$= \boxed{\iint_D \sqrt{4 + 2(u^2+v^2)} \, du\,dv}$$

double integral

use polar coordinates

$u = r\cos\theta$, $v = r\sin\theta$

$$= \int_0^{2\pi} d\theta \int_0^1 \sqrt{4+2r^2} \, r\,dr = \ldots$$

u-sub

EXAMPLE 6. *Find the surface area of the part paraboloid* $z = \overbrace{x^2 + y^2}^{f(x,y)}$ *between two planes:* $z = 0$ *and* $z = 4$.

Way 1 Special case

$z = f(x, y) = x^2 + y^2$
where $(x, y) \in D$ and
$D = \{(x, y) \mid x^2 + y^2 \le 4\}$

$$|\vec{N}(x,y)| = \sqrt{f_x^2 + f_y^2 + 1} =$$
$$= \sqrt{(2x)^2 + (2y)^2 + 1}$$
$$= \sqrt{4(x^2+y^2) + 1}$$

$$A(S) = \iint_S dS = \iint_D |\vec{N}(x,y)|\, dA$$

$$= \boxed{\iint_D \sqrt{4(x^2+y^2)+1}\, dA}$$

$$= \int_0^{2\pi} d\theta \int_0^2 \sqrt{4r^2+1}\, r\, dr =$$

$= u\text{-sub} = \ldots$

Way 2 Use cylindrical coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$z = x^2 + y^2 = r^2$ $\quad 0 \le z \le 4 \Rightarrow 0 \le r^2 \le 4$

$$\vec{R}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$$

$$D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

$$\vec{N}(r,\theta) = \vec{R}_r \times \vec{R}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} =$$

$$= \langle -2r^2\cos\theta, -(-2r^2\sin\theta), r \rangle$$

$$|\vec{N}(r,\theta)| = \sqrt{\underbrace{4r^4\cos^2\theta + 4r^4\sin^2\theta}_{} + r^2}$$

$$= \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$$

$$A(S) = \iint_S dS = \iint_D |\vec{N}(r,\theta)|\, dA =$$

$$= \int_0^{2\pi} d\theta \int_0^2 r\sqrt{4r^2+1}\, dr = $$

$u\text{-sub} \ldots$