16.7: Surface Integrals

Problem: Find the **mass** of a thin sheet (say, of aluminum foil) which has a shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$.

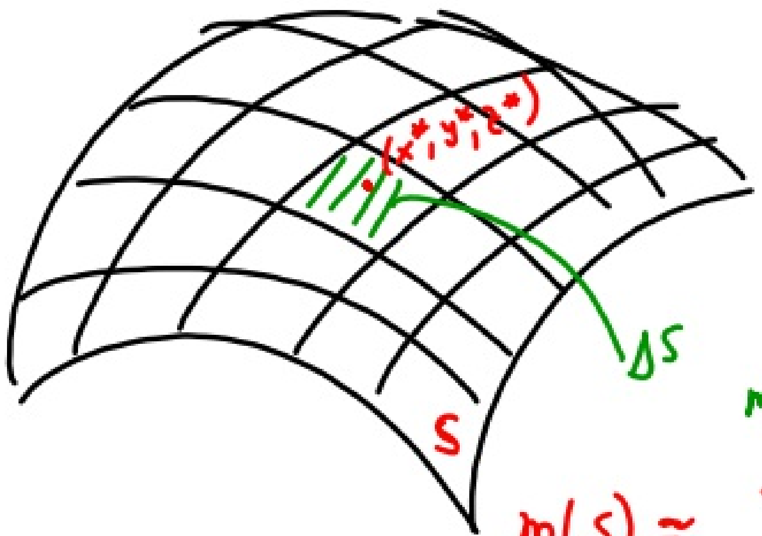
Solution:

$\rho = \text{const}$

$\Downarrow$

$$m = \rho \cdot SA$$

$$= \rho \iint_S dS$$



$$m(\Delta S) = \rho(x^*, y^*, z^*)\,\Delta S$$

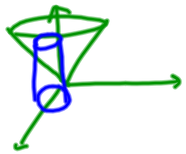
$$m(S) \approx \sum m(\Delta S) = \sum \rho\,\Delta S$$

$$m(S) = \iint_S \rho(x, y, z)\,dS$$

If S is given by $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, $(u,v) \in D$, then the **surface integral of f over the surface S** is:

$$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(\underline{\mathbf{r}(u,v)})|\mathbf{N}(u,v)|\,\mathrm{d}A = \iint_D f(x(u,v), y(u,v), z(u,v))|\vec{r}_u \times \vec{r}_v|\,dA$$

EXAMPLE 1. *Find the mass of a thin funnel in the shape of a cone* $\underbrace{z = \sqrt{x^2+y^2}}_{S:}$ *inside the cylinder* $\underbrace{x^2+y^2 \le 2x}_{(x-1)^2+y^2 \le 2x}$*, if its density is a function* $\rho(x,y,z) = x^2+y^2+z^2$.



Solution special case (x and y are parameters)

$$|\vec{N}(x,y)| = \sqrt{z_x^2 + z_y^2 + 1} =$$

$$= \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1} =$$

$$= \sqrt{\underbrace{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}_{1} + 1} = \sqrt{1+1} = \sqrt{2}$$

Parameters domain $D = \{(x,y) \mid x^2+y^2 \le 2x\}$

$z = \sqrt{x^2+y^2}$

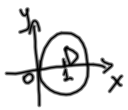
$$m = \iint_S \rho(x,y,z)\, dS = \iint_S (x^2+y^2+z^2)\, dS =$$

$$= \iint_D \left(x^2+y^2 + \left(\sqrt{x^2+y^2}\right)^2\right) |\vec{N}(x,y)|\, dA$$

$$= \iint 2(x^2+y^2)\sqrt{2}\, dA \underset{\text{use polar coordinates}}{=} \iint 2\sqrt{2}\, r^2\, r\, dA^*$$

$D = \{(x,y) \mid x^2+y^2 \le 2x\}$

$D^* = \{(r,\theta) \mid 0 \le r \le 2\cos\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$

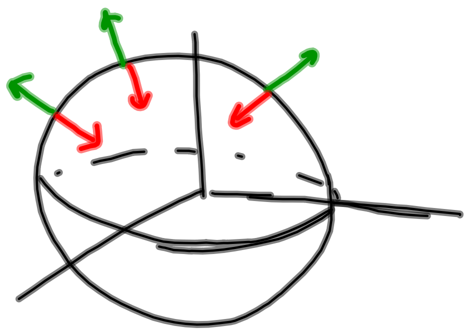


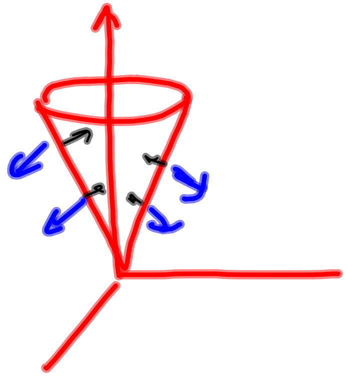
$$m = 2\sqrt{2} \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta$$

$$= 2\sqrt{2} \int_{-\pi/2}^{\pi/2} \frac{r^4}{4}\Big|_{r=0}^{2\cos\theta} d\theta = 2\sqrt{2} \int_{-\frac{\pi}{2}}^{\pi/2} 4\cos^4\theta\, d\theta = \ldots$$

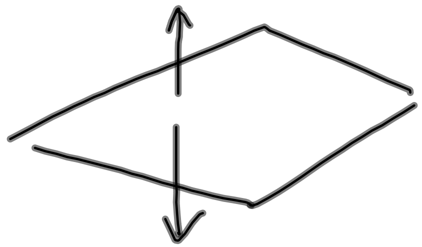
• **Oriented surfaces.** We consider only two-sided surfaces.

Let a surface S has a tangent plane at every point (except at any boundary points). There are two unit normal vectors at (x, y, z): $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$.





If it is possible to choose a unit normal vector $\hat{\mathbf{n}}$ at every point (x, y, z) of a surface S so that $\hat{\mathbf{n}}$ varies continuously over S, then S is called **oriented surface** and the given choice of $\hat{\mathbf{n}}$ provides S with an **orientation**. There are two possible orientations for any orientable surface:



`Convention:` *For closed surfaces the positive orientation is outward.*

for sphere

for cylinder with two lids

- **Surface integrals of vector fields.**

DEFINITION 2. *If* $\mathbf{F}$ *is a continuous vector field defined on an oriented surface* S *with unit normal vector* $\hat{\mathbf{n}}$*, then the* **surface integral of F over** S *is*

$$\iint_S \mathbf{F} \cdot d\vec{\mathbf{S}} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS.$$

This integral is also called the **flux** *of* $\mathbf{F}$ *across* S*.*

Note that if S is given by $\mathbf{r}(u, v)$, $(u, v) \in D$, then

$$\vec{n}(u,v) = \hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

and

$$d\vec{\mathbf{S}} = \hat{n}\, dS = \hat{n}(u,v) \cdot |\vec{n}(u,v)|\, du\, dv$$

$$= \frac{\vec{n}(u,v)}{|\vec{n}\cancel{(u,v)}|} \cdot |\vec{n}\cancel{(u,v)}|\, du\, dv = \vec{n}(u,v)\, du\, dv$$

$$= \vec{r}_u \times \vec{r}_v\, du\, dv$$

Finally,

dot product.

make sure to choose correct sign for normal

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

EXAMPLE 3. *Find the flux of the vector field*

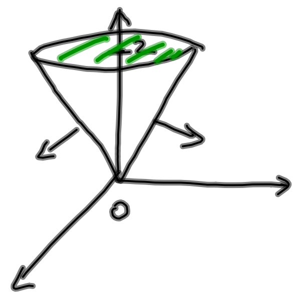
$$\mathbf{F} = \langle x^2, y^2, z^2 \rangle$$

across the surface $(x,y)|$

$$S = \{z^2 = x^2 + y^2, 0 \le z \le 2\}.$$

normal is pointed out of S.

$$S = \{ \overset{(x,y)|}{z} = \sqrt{x^2+y^2},\ 0 \le z \le 2 \}$$



$0 \le z = \sqrt{x^2+y^2} \le 2$

$x^2+y^2 \le 4$

Solution Parameterize S:

$$\vec{r}(x,y) = \langle x, y, \underbrace{\sqrt{x^2+y^2}}_{z} \rangle$$

$$D = \{ (x,y) \mid x^2+y^2 \le 4 \}$$

$$\vec{n}(x,y) = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix} =$$

$$= \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1 \right\rangle$$

Flux

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(x,y))\cdot \vec{n}(x,y)\, \overbrace{dx\,dy}^{dA}$$

$$= \iint_D \langle x^2, y^2, \left(\sqrt{x^2+y^2}\right)^2\rangle \cdot \vec{n}(x,y)\,dx\,dy$$

dot product

$$= \iint_D \langle x^2, y^2, x^2+y^2\rangle \cdot \langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\rangle\,dx\,dy$$

$$= \iint_{D=\{(x,y)\mid x^2+y^2\le 4\}} \left(\frac{\overbrace{-x^3-y^3}^{-(x^3+y^3)}}{\sqrt{x^2+y^2}} + x^2+y^2\right)dx\,dy \overset{\text{use polar coordinates}}{=}$$

$$= \int_0^{2\pi} d\theta \int_0^2 \left(\frac{-r^3(\cos^3\theta+\sin^3\theta)}{r} + r^2\right) r\,dr =$$

$$= \int_0^{2\pi}\int_0^2 \left(r^3\cos^3\theta + r^3\sin^3\theta - r^3\right)dr\,d\theta$$

$$\underbrace{\int_0^{2\pi}(\cos^3\theta+\sin^3\theta-1)\,d\theta}_{-2\pi} \int_0^2 r^3\,dr = -2\pi \left.\frac{r^4}{4}\right|_0^2 = -\frac{\pi}{2}\cdot 16 = \boxed{-8\pi}$$

because $\int_0^{2\pi}\cos^3\theta\,d\theta = \int_0^{2\pi}\sin^3\theta\,d\theta = 0$