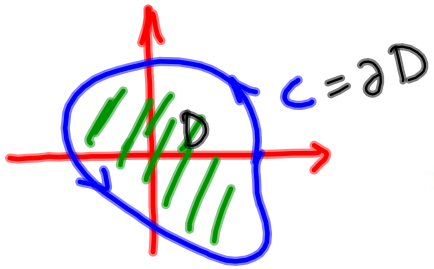
$$\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\hat{k}$$

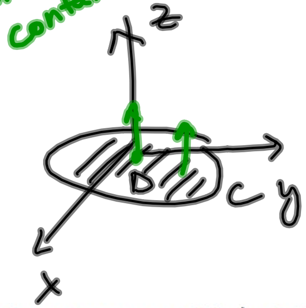
16.8: STOKES' THEOREM

Stokes' Theorem can be regarded as a 3-dimensional version of Green's Theorem:

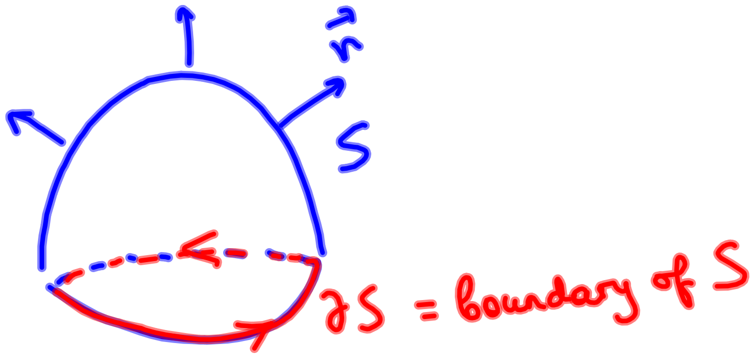
normal to the plane containing D (z=0)

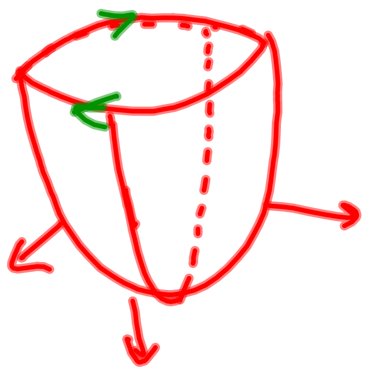


$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}A = \iint_D \operatorname{curl}\mathbf{F}\cdot \mathbf{k}\,\mathrm{d}A.$$



Let S be an oriented surface with unit normal vector $\hat{\mathbf{n}}$ and with the boundary curve C (which is a space curve).





The orientation on S induces the **positive orientation of the boundary curve** C: if you walk in the positive direction around C with your head pointing in the direction of $\hat{\mathbf{n}}$, then the surface will always be on your left.

The positively oriented boundary curve of an oriented surface S is often written as ∂S.

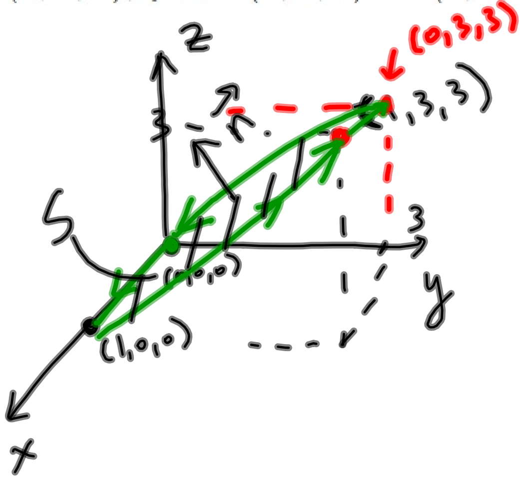
Stokes' Theorem: *Let S be an oriented piece-wise-smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \mathrm{curl}\mathbf{F} \cdot \mathrm{d}\mathbf{S},$$

or

$$\iint_S \mathrm{curl}\mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

EXAMPLE 1. *Find the work performed by the forced field* $\mathbf{F}(x,y,z) = \langle 3x^8, 4xy^3, y^2x\rangle$ *on a particle that traverses the curve* C *in the plane* $z = y$ *consisting of 4 line segments from* $(0,0,0)$ *to* $(1,0,0)$*, from* $(1,0,0)$ *to* $(1,3,3)$*, from* $(1,3,3)$ *to* $(0,3,3)$*, and from* $(0,3,3)$ *to* $(0,0,0)$.



$$W = \oint_C \vec{F}\cdot d\vec{r}$$

Way 1 Parameterize four segments that give the curve C and calculate four line integrals.

Way 2 Note that C is closed and hence we can apply Stokes' theorem. To that end, we choose the surface S as a part of the plane z=y bounded by C.

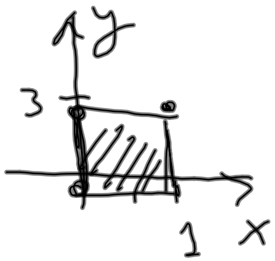
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^8 & 4xy^3 & y^2 x \end{vmatrix}$$

$$= \langle 2yx - 0,\ -(y^2 - 0),\ 4y^3 - 0 \rangle$$

$$= \langle 2yx,\ -y^2,\ 4y^3 \rangle$$

Parameterize S (special case)



$x = x, \quad y = y, \quad z = y$

(or $\vec{r}(x,y) = \langle x, y, y \rangle$)

$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 3\}$

projection onto the xy-plane

To find normal

way 1

$\vec{n}(x,y) = \pm \vec{r}_x \times \vec{r}_y = \ldots$

..

way2 (a shorter one)

$\vec{n}(x,y) = \vec{n} = \pm\langle z_x, z_y, -1 \rangle = \pm \langle 0, 1, -1 \rangle$

Because the z-component of $\vec{n}$ must be positive

$\vec{n}(x,y) = \langle 0, -1, 1 \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} =$$

$$= \iint_D \langle 2yx, -y^2, 4y^3 \rangle \cdot \vec{n}(x,y)\, dA_{xy}$$

dot product

$$= \iint_D \langle 2yx, -y^2, 4y^3 \rangle \cdot \langle 0, -1, 1 \rangle\, dA_{xy}$$

$$= \int_0^1 \int_0^3 \left(2yx \cdot 0 + (-y^2) \cdot (-1) + 4y^3 \cdot 1\right) dy\,dx =$$

$$= \int_0^1 \left(\int_0^3 (y^2 + 4y^3)\, dy \right) dx$$

$$= \left(\frac{y^3}{3} + y^4 \right) \Big|_0^3 = 9 + 81 = \boxed{90}$$

EXAMPLE 2. *Verify Stokes' Theorem* $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S}\vec{F}\cdot d\vec{r}$ *for the vector field* $\vec{F} = \langle 3y, 4z, -6x\rangle$ *and the paraboloid* $z = 9 - x^2 - y^2$ *that lies above the plane* $z = -7$ *and oriented upward. Be sure to check and explain the orientations.*

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$C = \partial S$

Solution: Use the following steps:

• Parametrize the boundary circle ∂S and compute the line integral.

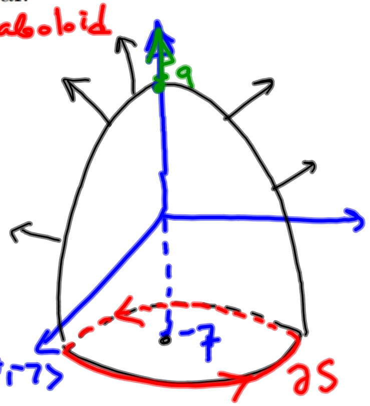
∂S is the line of intersection of the paraboloid $z = 9 - x^2 - y^2$ and the plane $z = -7$

$$\Rightarrow -7 = 9 - x^2 - y^2$$
$$-7 = 9 - (x^2 + y^2)$$

$$\partial S \quad \boxed{z = -7, \; x^2 + y^2 = 16}$$

∂S: $x = 4\cos\theta$, $y = 4\sin\theta$, $z = -7$, $0 \le \theta \le 2\pi$

or $\vec{r}(\theta) = \langle 4\cos\theta, 4\sin\theta, -7\rangle$



$$\oint_{\partial S} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(\theta))\cdot\vec{r}\,'(\theta)\,d\theta =$$

$\vec{F} = \langle 3y, 4z, -6x\rangle$

$$= \int_0^{2\pi} \langle 3\cdot 4\sin\theta, 4\cdot(-7), -6\cdot 4\cos\theta\rangle \cdot \langle -4\sin\theta, 4\cos\theta, 0\rangle\,d\theta =$$

$$= \int_0^{2\pi} \left(-48\sin^2\theta - 28\cdot 4\cos\theta + 0\right)d\theta$$

$$= \int_0^{2\pi} -48\cdot\frac{1-\cos 2\theta}{2} = -24\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi}$$

$$= -24\cdot 2\pi = \boxed{-48\pi}$$

$$\int_0^{2\pi} \cos\theta = 0$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

•Parametrize the surface of the paraboloid and compute the surface integral:

$S = \{ (x,y,z) \mid z = 9 - x^2 - y^2 ,\quad z \geq -7 \}$

$9 - x^2 - y^2 \geq -7$

$\vec{r}\,(x,y) = \langle x, y, 9 - x^2 - y^2 \rangle$

$D = \{ (x,y) \mid x^2 + y^2 \leq 16 \}$

$\vec{n}\,(x,y) = \pm \langle z_x, z_y, -1 \rangle$

$= \pm \langle -2x, -2y, -1 \rangle$

$\vec{n}\,(x,y) = \langle 2x, 2y, 1 \rangle$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 4z & -6x \end{vmatrix} =$$

$= \langle 0-4, -(-6-0), 0-3 \rangle = \langle -4, 6, -3 \rangle$

Title : Apr 24-11:05 AM (Page 7 of 9)

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_D \text{curl}\,\vec{F}(\vec{r}(x,y))\cdot\vec{n}(x,y)\,dA_{xy}$$

$$= \iint_D \langle -4, 6, -3\rangle \cdot \langle 2x, 2y, 1\rangle\, dA$$

use polar coordinates

$$= \iint_D (-8x + 12y - 3)\, dA =$$

$$= \int_0^{2\pi}\int_0^4 (-8r\cos\theta + 12r\sin\theta - 3)\,r\,dr\,d\theta$$

$$= \underbrace{\left(\int_0^{2\pi}\cos\theta\,d\theta\right)}_{=0}\int_0^4 -8r^2\,dr + \overbrace{\left(\int_0^{2\pi}\sin\theta\,d\theta\right)}^{=0}\int_0^4 12r^2\,dr$$

$$-3\int_0^{2\pi} d\theta \int_0^4 r\,dr = -3\cdot 2\pi\, \frac{r^2}{2}\Big|_0^4 = \boxed{-48\pi}$$

Conclusion: $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\,d\vec{S} = -48\pi.$

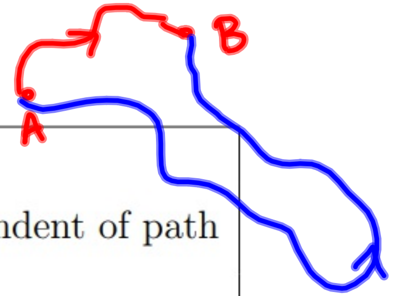
THEOREM 3. *If* $\mathbf{F}$ *is a vector field defined on* $\mathbb{R}^3$ *whose component functions have continuous partial derivatives and* $\operatorname{curl}\mathbf{F} = \mathbf{0}$, *then* $\mathbf{F}$ *is a conservative vector field.*

SUMMARY: Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a continuous vector field *in* $\mathbb{R}^3$.

potential function

There exists f s.t. $\nabla f = \mathbf{F}$

$\int_{\breve{AB}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path



$\mathbf{F}$ is conservative in $\mathbb{R}^3$

$\operatorname{curl}\mathbf{F} = \mathbf{0}$

$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C