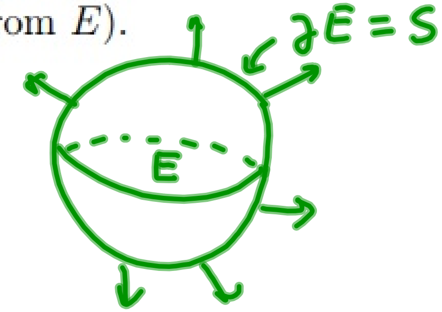
16.9: The Divergence Theorem

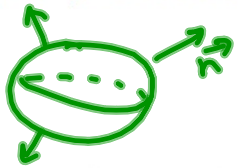
Let E be a simple solid region with the boundary surface S (which is a closed surface.) Let S be positively oriented (i.e.the orientation on S is outward that is, the unit normal vector $\hat{\mathbf{n}}$ is directed outward from E).



		Boundary
Green theorem	plain region D	∂D is a plane curve
Stokes' Theorem	surface S	∂S is a curve (or union of two or more curves)
Divergence theorem	solid E	∂E is a surface (closed)

D

The Divergence Theorem: Let E be a simple solid region whose boundary surface S has positive (outward) orientation. Let $\mathbf{F}$ be a continuous vector field on an open region that contains E. Then



$$= \oiint_S \vec{F}\cdot\hat{n}\, dS$$

$$\underbrace{\oiint_S \mathbf{F}\cdot d\mathbf{S}}_{\text{Flux}} = \iiint_E \underbrace{\operatorname{div}\mathbf{F}}_{\text{scalar field}}\, dV.$$

Divergence is the tendency of the vector field to *diverge from/to move toward* the point ($\operatorname{div}\mathbf{F} > 0$ corresponds to expansion; $\operatorname{div}\mathbf{F} < 0$ corresponds to compression).

In $\operatorname{div}\mathbf{F} = const$, then

$$\oiint_S \vec{F}\cdot d\vec{S} = \operatorname{div}\vec{F} \iiint_E dV$$

$$\text{const} = \operatorname{div}\vec{F} = \frac{\oiint_S \vec{F}\cdot d\vec{S}}{\iiint_E dV} = \frac{\text{Flux through } S = \text{boundary of } E}{\text{Volume }(E)}$$

The Divergence Theorem says that the divergence is the outgoing/ingoing flux per volume.

EXAMPLE 1. *Let* $E = \{(x,y,z) : x^2+y^2 \le R^2, 0 \le z \le H\}$. *Find the flux of the vector field* $\mathbf{F} = \langle \underbrace{1+x}_{P}, \underbrace{3+y}_{Q}, \underbrace{z-10}_{R} \rangle$ *over* ∂E.

$$\underbrace{\iint_{\partial E}}_{\text{closed surface}} \vec{F}\cdot d\vec{S}$$

To calculate this flux directly we should parameterize 3 surfaces

$\Downarrow$

Apply Divergence theorem

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\text{Flux over } \partial E = \iint_{\partial E} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV =$$

$$= \iiint_E (1+1+1)\, dV = 3\iiint_E dV = 3V(E) = 3\pi R^2 H.$$

REMARK 2. If $\mathbf{F} = \left\langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3} \right\rangle$ then $= \operatorname{div}\vec{F} = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$

$$\oiint_{\partial E} \vec{F}\cdot d\vec{S} = \underbrace{\iiint_E \underbrace{\operatorname{div}\vec{F}}_{=1}\, dV}_{\mathrm{Vol}(E)}$$

$$\boxed{\mathrm{Vol}(E) = \oiint_{\partial E} \left\langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3} \right\rangle \cdot d\vec{S}}$$

EXAMPLE 3. *Evaluate* $I = \oiint_S \underbrace{\text{curl}\mathbf{F}}_{\text{vector field}} \cdot d\mathbf{S}$ *if* S *is the boundary of*

(a) *ellipsoid* $E = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\}$ *and* $\mathbf{F} = \langle$ ●, ☺, ❄ $\rangle$

solid

$$I = \oiint_S \text{curl}\,\vec{F}\cdot d\vec{S} \overset{\text{Divergence Theorem}}{=} \iiint_E \underbrace{\text{div}\,(\text{curl}\,\vec{F})}_{=0}\, dV = 0$$

↑ closed surface

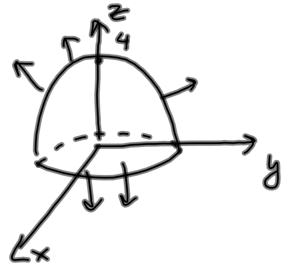
b/c S is a boundary of solid

(b) *an arbitrary simple solid region* E *and* $\vec{F}$ *is an arbitrary continuous vector field.*

then

$$\oiint_{S=\partial E} \text{curl}\,\vec{F}\cdot d\vec{S} = 0.$$

EXAMPLE 4. *Let E be the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Evaluate* $I = \iint_S \langle x^3, 2xz^2, 3y^2 z\rangle \cdot d\mathbf{S}$ *if*



(a) *S is the boundary of the solid E.*

$\Downarrow$

S is a closed surface

Thus we apply D.T.

$$I = \underset{S}{\oiint} \langle x^3, 2xz^2, 3y^2 z\rangle \cdot dS \overset{DT}{=} \iiint_E \operatorname{div} \vec{F}\, dV =$$

$$= \iiint_E (3x^2 + 0 + 3y^2)\, dV = 3 \iiint_E (x^2+y^2)\, dV =$$

$$= 3 \iint_D \left[\int_0^{4-(x^2+y^2)} (x^2+y^2)\, dz \right] dA$$

where D is projection of E onto the xy-plane:

$$D = \{(x,y) \mid x^2+y^2 \le 4\}$$

$$I = 3 \iint_D (x^2+y^2)\, z \Big|_{z=0}^{4-(x^2+y^2)} dA$$

$$= 3 \iint_D (x^2+y^2)\left(4-(x^2+y^2)\right) dA$$

use polar coordinates:

$x = r\cos\theta$, $y = r\sin\theta$

$$= 3 \int_0^{2\pi} \int_0^{2} r^2(4-r^2)\, r\, dr\, d\theta$$

$$= 3 \cdot 2\pi \int_0^2 (4r^3 - r^5)\, dr$$

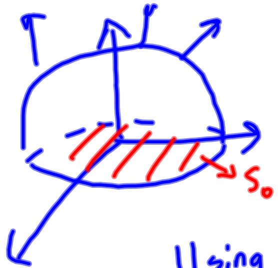
$$= 6\pi \left(r^4 - \frac{r^6}{6} \right) \Big|_0^2$$

$$= 6\pi \left(16 - \frac{64}{6} \right)$$

$$= 96\pi - 64\pi = 32\pi$$

(B) $\hat{S}$ *is the part of the* $\underbrace{\text{paraboloid } z = 4 - x^2 - y^2}_{\text{surface}}$ *between the planes* $z = 0$ *and* $z = 4$.

$\hat{S}$ is not closed surface



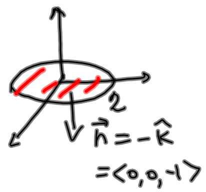
$$\iint_{\hat{S}} \vec{F} \cdot d\vec{S}$$

Using part (A) we get

$$32\pi = \iint_{\substack{S=\partial E \\ \uparrow \\ \text{part A}}} \vec{F} \cdot d\vec{S} = \boxed{\iint_{\hat{S}} \vec{F} \cdot d\vec{S}} + \iint_{S_0} \vec{F} \cdot d\vec{S}$$

S_0,

$$\iint_{\hat{S}} \vec{F} \cdot d\vec{S} = 32\pi - \iint_{S_0} \vec{F} \cdot \underbrace{d\vec{S}}_{\vec{n}\,dS}$$



Parameterize $S_0 = \{(x,y,z) \mid x^2 + y^2 \le 4,\ z = 0\}$

$x = x, \quad y = y, \quad z = 0$

$D = \{(x,y) \mid x^2 + y^2 \le 4\}$

$$\iint_{\hat{S}} \vec{F} \cdot d\vec{S} = 32\pi - \iint_D \langle x^3, 2x \cdot 0^2, 3y^2 \cdot 0 \rangle \cdot \overset{\text{normal}}{\langle 0, 0, -1 \rangle} \cdot dA$$

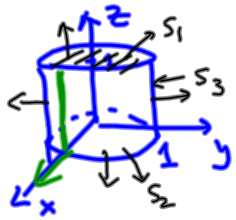
$$= 32\pi - 0 = 32\pi.$$

EXAMPLE 5. *Verify the Divergence Theorem for the vector field* $\mathbf{F}(x,y,z) = \langle x^3, y^3, z^3\rangle$ *and* S, *which is the surface of the region enclosed by the cylinder* $x^2+y^2=1$ *and the planes* $z=1$ *and* $z=0$.

Divergence Theorem:

$$\underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{\text{LHS}} = \overbrace{\iiint_E \operatorname{div}\vec{F}\, dV}^{\text{RHS}},$$

where $E = \{(x,y,z) \mid x^2+y^2 \le 1,\ 0 \le z \le 1\}$



$S = \overset{\text{top}}{S_1} \cup \overset{\text{bottom}}{S_2} \cup \overset{\text{sides}}{S_3}$

LHS

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S}$$

Parameterize $S_1 = \{(x,y,z) \mid x^2+y^2 \le 1,\ z=1\}$

$x=x,\ y=y,\ z=1$, $D_1 = \{(x,y) \mid x^2+y^2 \le 1\}$

$\vec{n}(x,y) = \langle 0,0,1\rangle$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} \langle x^3, y^3, z^3\rangle \cdot d\vec{S} = \iint_{D_1} \langle x^3, y^3, 1^3\rangle \cdot \langle 0,0,1\rangle\, dA = \iint_{D_1} dA = A(D_1) = \boxed{\pi}$$

Parameterize $S_2 = \{(x,y) \mid x^2+y^2 \le 1,\ z=0\}$

$x=x,\ y=y,\ z=0$, $D_2 = \{(x,y) \mid x^2+y^2 \le 1\}$

$\vec{n}(x,y) = \langle 0,0,-1\rangle$

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_{S_2} \langle x^3, y^3, z^3\rangle \cdot d\vec{S} = \iint_{D_2} \langle x^3, y^3, 0\rangle \langle 0,0,-1\rangle\, dA = \boxed{0}.$$

Parametrize $S_3 = \{(x,y,z) \mid x^2+y^2=1,\ 0 \le z \le 1\}$

Use cylindrical coordinates:

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

For S_3: $x = \cos\theta$, $y = \sin\theta$, $z = z$ OR

$\vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$, $D_3 = \{(\theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le z \le 1\}$

Find normal to S_3:

$$\vec{n}(\theta, z) = \pm\, \vec{r}_\theta \times \vec{r}_z = \pm \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$= \pm \langle \cos\theta, \sin\theta, 0 \rangle$$

If $\theta = 0 \Rightarrow \vec{n}$ should be in the direction of $\hat{i}$

$$\iint_{S_3} \vec{F}\cdot d\vec{S} = \iint_{S_3} \langle x^3, y^3, z^3 \rangle \cdot d\vec{S} =$$

$$= \iint_{D_3} \langle \cos^3\theta, \sin^3\theta, z^3 \rangle \cdot \langle \cos\theta, \sin\theta, 0 \rangle\, dA$$

$$= \iint_{D_3} (\cos^4\theta + \sin^4\theta)\, dA = \int_0^{2\pi}\int_0^1 (\cos^4\theta + \sin^4\theta)\, dz\, d\theta \ldots$$

$$= 2\int_0^{2\pi} \cos^4\theta\, d\theta = 2 \int^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta =$$

$$= \frac{1}{2}\left(\int_0^{2\pi} 1^2\, d\theta + \underbrace{2\int_0^{2\pi} \cos 2\theta\, d\theta}_{0} + \int_0^{2\pi} \cos^2 2\theta\, d\theta \right) =$$

$$= \frac{1}{2}\left(2\pi + \int_0^{2\pi} \frac{1+\cos 4\theta}{2}\, d\theta\right) = \frac{1}{2}\left(2\pi + \underbrace{\int_0^{2\pi} \frac{d\theta}{2}}_{\pi} + \underbrace{\int_0^{2\pi} \frac{\cos 4\theta\, d\theta}{2}}_{0}\right) =$$

$$= \boxed{3\pi/2}$$

$$\text{LHS} = \pi + 0 + \frac{3\pi}{2} = \frac{5\pi}{2}$$

RHS

$$\iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E \operatorname{div} \langle x^3, y^3, z^3 \rangle\, dV$$

$$= \iiint_E (3x^2 + 3y^2 + 3z^2)\, dV =$$

Use cylindrical coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$E = \{(x,y,z) \mid x^2 + y^2 \le 1,\ 0 \le z \le 1\}$

$E^* = \{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1\}$

$$\iiint_E \operatorname{div} \vec{F}\, dV = \iiint_{E^*} 3(r^2 + z^2)\, r\, dV^*$$

$$= 3\int_0^{2\pi}\int_0^1\int_0^1 (r^2 + z^2)\, r\, dr\, dz\, d\theta = \ldots$$

$$= 3\cdot 2\pi \int_0^1\int_0^1 (r^3 + rz^2)\, dr\, dz$$

$$= 6\pi \int_0^1 \left(\frac{r^4}{4} + \frac{r^2z^2}{2}\right)\Big|_0^1 dz = 6\pi\int_0^1 \left(\frac{1}{4} + \frac{z^2}{2}\right) dz$$

$$= 6\pi\left(\frac{z}{4} + \frac{z^3}{6}\right)\Big|_0^1 = \pi\left(\frac{3z}{2} + z^3\right)\Big|_0^1$$

$$= \pi\left(\frac{3}{2} + 1\right) = \frac{5\pi}{2}$$

LHS = RHS