11.5: Quadric surfaces

REVIEW: Parabola, hyperbola and ellipse.

- Parabola: \( y = ax^2 \) or \( x = ay^2 \).

- Ellipse: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).
  Intercepts: \((\pm a, 0)\) & \((0, \pm b)\)

- Hyperbola: \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) or \( -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).
  Intercepts: \((\pm a, 0)\) & \((0, \pm b)\)
The most general second-degree equation in three variables $x, y$ and $z$:

$$Ax^2 + By^2 + Cz^2 + axy + bxz + cyz + d_1x + d_2y + d_3z + E = 0,$$

where $A, B, C, a, b, c, d_1, d_2, d_3, E$ are constants. The graph of (1) is a quadric surface.

Note if $A = B = C = a = b = c = 0$ then (1) is a linear equation and its graph is a plane (this is the case of degenerated quadric surface).

By translations and rotations (1) can be brought into one of the two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \text{ or } Ax^2 + By^2 + Iz = 0.$$

In order to sketch the graph of a surface determine the curves of intersection of the surface with planes parallel to the coordinate planes. The obtained in this way curves are called traces or cross-sections of the surface.

Quadric surfaces can be classified into 5 categories:

**ellipsoids, hyperboloids, cones, paraboloids, quadric cylinders.** (shown in the table, see Appendix.)

The elements which characterize each of these categories:

1. Standard equation.
2. Traces (horizontal (by planes $z = k$), $yz$-traces (by $x = 0$) and $xz$-traces (by $y = 0$).
3. Intercepts (in some cases).

**To find the equation of a trace** substitute the equation of the plane into the equation of the surface (cf. Example 4, Section 1.1 notes). Note, in the examples below the constants $a, b, and c$ are assumed to be positive.
TECHNIQUES FOR GRAPHING QUADRATIC SURFACES

- **Ellipsoid.** Standard equation:

  \[
  \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
  \]

  Note if \(a = b = c\) we have a \underline{sphere}.

EXAMPLE 1. Sketch the ellipsoid

\[
\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1
\]

Solution

- Find \textbf{intercepts}:
  * \(x\)-intercepts: if \(y = z = 0\) then \(x = \)
  * \(y\)-intercepts: if \(x = z = 0\) then \(y = \)
  * \(z\)-intercepts: if \(x = y = 0\) then \(z = \)

- Obtain \textbf{traces of}:
  * the \(xy\)-plane: plug in \(z = 0\) and get \(\frac{x^2}{9} + \frac{y^2}{16} = 1\)

  * the \(yz\)-plane: plug in \(x = 0\) and get

  * the \(xz\)-plane: plug in \(y = 0\) and get
- Hyperboloids: There are two types:
  - Hyperboloid of one sheet.
    Standard equation:
    \[
    \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1
    \]
    
    EXAMPLE 2. Sketch the hyperboloid of one sheet
    \[
    x^2 + y^2 - \frac{z^2}{9} = 1
    \]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Trace</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z = 0)</td>
<td>(z = \pm 3)</td>
</tr>
<tr>
<td>(x = 0)</td>
<td>(y = 0)</td>
</tr>
</tbody>
</table>
– Hyperboloid of two sheets.

Standard equation:

\[-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\]

EXAMPLE 3. Sketch the hyperboloid of two sheet

\[-x^2 - \frac{y^2}{9} + z^2 = 1\]

Solution Find z-intercepts: if \(x = y = 0\) then \(z = \)

<table>
<thead>
<tr>
<th>Plane</th>
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</tr>
</thead>
<tbody>
<tr>
<td>(z = \pm 2)</td>
<td></td>
</tr>
<tr>
<td>(x = 0)</td>
<td></td>
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<tr>
<td>(y = 0)</td>
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</tbody>
</table>
Elliptic Cones. Standard equation:

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \]

If \( a = b = c \) then we say that we have a circular cone.

EXAMPLE 4. Sketch the elliptic cone

\[ z^2 = x^2 + \frac{y^2}{9} \]

<table>
<thead>
<tr>
<th>Plane</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( z = \pm 1 )</td>
<td></td>
</tr>
<tr>
<td>( x = 0 )</td>
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<tr>
<td>( y = 0 )</td>
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</tbody>
</table>

Special cases:

1. \( a = b = c \)
2. \( z = \sqrt{x^2 + y^2} \)
3. \( z = -\sqrt{x^2 + y^2} \)
• **Paraboloids** There are two types:
  
  - **Elliptic paraboloid.** Standard equation:
    \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} \]

EXAMPLE 5. *Sketch the elliptic paraboloid*

\[ z = \frac{x^2}{4} + \frac{y^2}{9} \]

<table>
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<tbody>
<tr>
<td>(z = 1)</td>
<td></td>
</tr>
<tr>
<td>(x = 0)</td>
<td></td>
</tr>
<tr>
<td>(y = 0)</td>
<td></td>
</tr>
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</table>

Special case: \(a = b\)
Hyperbolic paraboloid. Standard equation:

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}
\]

If \( z = k \) then

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{k}{c}
\]

**EXAMPLE 6. Sketch the hyperbolic paraboloid**

\[
z^2 = x^2 - y^2
\]

<table>
<thead>
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</thead>
<tbody>
<tr>
<td>( z = 1 )</td>
<td></td>
</tr>
<tr>
<td>( z = -1 )</td>
<td></td>
</tr>
<tr>
<td>( x = 0 )</td>
<td></td>
</tr>
<tr>
<td>( y = 0 )</td>
<td></td>
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</table>
• Quadric cylinders: There are three types:

**Elliptic cylinder:**
- Standard equation:
  \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

**EXAMPLE 7.** Sketch elliptic cylinder

\[ x^2 + \frac{y^2}{4} = 1 \]

**Hyperbolic cylinder:**
- Standard equation:
  \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

**EXAMPLE 8.** Sketch hyperbolic cylinder

\[ x^2 - y^2 = 1 \]

**Parabolic cylinder:**
- Standard equation:
  \[ y = ax^2 \]

**EXAMPLE 9.** Sketch parabolic cylinder

\[ y = -x^2 \]
CONCLUSION

<table>
<thead>
<tr>
<th></th>
<th>Equation</th>
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<tbody>
<tr>
<td>Ellipsoid</td>
<td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$</td>
</tr>
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<td>Parabolic cylinder</td>
<td>$y = ax^2$</td>
</tr>
</tbody>
</table>

TRANSLATIONS AND REFLECTIONS OF QUADRIC SURFACES

EXAMPLE 10. Describe and sketch the surface $z = (x + 4)^2 + (y - 2)^2 + 5$. 
Note that replacing a variable by its negative in the equation of a surface causes that surface to be reflected about a coordinate plane.

EXAMPLE 11. Identify and sketch the surface.
(a) \( z = -(x^2 + y^2) \)

(b) \( y^2 = x^2 + z^2 \)
EXAMPLE 12. Classify and sketch the surface

\[ x^2 + y^2 + z - 4x - 6y + 13 = 0. \]