

12.6: Directional Derivatives and the Gradient Vector

Recall that the two partial derivatives $f_x(x, y)$ and $f_y(x, y)$ of $f(x, y)$ represent the rate of change of f as we vary x (holding y fixed) and as we vary y (holding x fixed) respectively. In other words, $f_x(x, y)$ and $f_y(x, y)$ represent the rate of change of f in the directions of the unit vectors \mathbf{i} and \mathbf{j} respectively. Let's consider how to find the rate of change of f if we allow both x and y to change simultaneously, or how to find the rate of change of f in the direction of an arbitrary vector \mathbf{u} .

DEFINITION 1. *The rate of change of $f(x, y)$ in the direction of the unit vector $\hat{\mathbf{u}} = \langle a, b \rangle$ is called the **directional derivative** and it is denoted by $D_{\mathbf{u}}f(x, y)$.*

*The **directional derivative** of f at (x_0, y_0) in the direction of the unit vector $\hat{\mathbf{u}} = \langle a, b \rangle$ is*

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

REMARK 2. By comparing the last definition with the definitions of the partial derivatives:

$$f_x(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}, \quad f_y(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

we see that

$$f_x(x_0, y_0) = \quad \quad \quad \text{and} \quad \quad \quad f_y(x_0, y_0) =$$

For computational purposes use the following theorem.

THEOREM 3. *If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\hat{\mathbf{u}} = \langle a, b \rangle$ and*

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

EXAMPLE 4. *Find the rate of change $f(x, y) = x^3 + \sin(xy)$ at the point $(1, \pi/2)$ in the direction indicated by the angle $\theta = \pi/4$.*

The Directional Derivative As The Dot Product Of Two Vectors. Gradient.

DEFINITION 5. The **gradient** of $f(x, y)$ is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}.$$

Notations for gradient: **grad** f or ∇f which is read "del f ".

EXAMPLE 6. Find the gradient of $f = \cos(xy) + e^x$ at $(0, 3)$.

By Theorem 3 we have:

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b =$$

Formula for the directional derivative using the gradient vector:

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \hat{\mathbf{u}}.$$

EXAMPLE 7. Find the directional derivative for f from Example 6 at $(0, 3)$ in the direction $\langle 3, 4 \rangle$.

The directional derivative of function of *three* variables

THEOREM 8. If f is a differentiable function of x , y and z , then f has a directional derivative in the direction of any unit vector $\hat{\mathbf{u}} = \langle a, b, c \rangle$ and

$$D_{\mathbf{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c = \nabla f \cdot \hat{\mathbf{u}},$$

where

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

is the gradient vector of $f(x, y, z)$.

EXAMPLE 9. Find the directional derivative of $f(x, y, z) = z^3 - x^2y$ at the point $(1, 6, 2)$ in the direction $\mathbf{u} = \langle 1, -2, 3 \rangle$.

QUESTION: In which of all possible directions does f change fastest and what is the maximum rate of change.

ANSWER is provided by the following theorem:

THEOREM 10. Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occurs when \mathbf{u} has the same direction as the gradient vector ∇f .

Proof.

EXAMPLE 11. Suppose that the temperature at a point (x, y, z) in the space is given by

$$T(x, y, z) = \frac{100}{1 + x^2 + y^2 + z^2},$$

where T is measured in $^{\circ}\text{C}$ and x, y, z in meters.

(a) In which direction does the temperature increase fastest at the point $(1, 1, -1)$?

(b) What is the maximum rate of increase?

Tangent planes to level surfaces:

FACT: The gradient vector $\nabla F(x_0, y_0, z_0)$ is **orthogonal** to the level surface $F(x, y, z) = k$ at the point (x_0, y_0, z_0) .

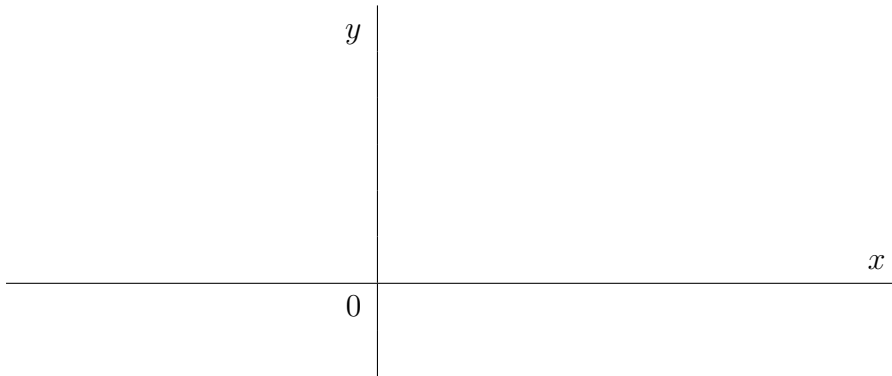
So, the *tangent plane* to the surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0) has the equation:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The normal line to the surface at the point (x_0, y_0, z_0) is the line passing through (x_0, y_0, z_0) and perpendicular to the tangent plane. Therefore its direction is given by the _____ vector

EXAMPLE 12. Find the equation of the tangent plane and normal line at the point $(1, 0, 5)$ to the surface $xe^{yz} = 1$.

Likewise, the gradient vector $\nabla f(x_0, y_0)$ is **orthogonal** to the level curve $f(x, y) = k$ at the point (x_0, y_0) .



Consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y) . Draw a curve of steepest ascent.