14.3: The fundamental Theorem for Line Integrals

14.4: Green's Theorem

- **Conservative vector field.**

DEFINITION 1. *A vector field* $\mathbf{F}$ *is called a* **conservative vector field** *if it is the gradient of some scalar function* f *s.t* $\mathbf{F} = \nabla f$. *In this situation* f *is called a* **potential function** *for* $\mathbf{F}$.

REMARK 2. Not all vector fields are conservative, but such fields do arise frequently in Physics.

`Illustration:` **Gravitational Field:** By Newton's Law of Gravitation the magnitude of the gravitational force between two objects with masses m and M is The gravitational force acting on the object at (x, y, z) is

$$|\mathbf{F}| = G\frac{mM}{r^2},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the objects and G is the gravitational constant.

Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z\rangle$. Then

$$r =$$

Then the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z\rangle$ is

$$\mathbf{F}(x, y, z) =$$

EXAMPLE 3. *Let*

$$f(x, y, z) = \frac{GmM}{\sqrt{x^2 + y^2 + z^2}}.$$

Find its gradient and answer the questions:

(a) *Is the gravitational field conservative?*

(b) *What is a potential function of the gravitational field?*

• **The fundamental Theorem for Line Integrals:** Recall Part 2 of the Fundamental Theorem of Calculus:

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a),$$

where F' is continuous on $[a, b]$.

Let C be a smooth curve given by $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables and ∇f is continuous on C. Then

$$\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof.

REMARK 4. If C is a closed curve then

COROLLARY 5. *If F is a conservative vector field and C is a curve with initial point A and terminal point B then:*

EXAMPLE 6. *Find the work done by the gravitational field*

$$\mathbf{F}(x,y,z) = -\frac{GmM}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle$$

in moving a particle with mass m from the point $(1,2,2)$ to the point $(3,4,12)$ along a piecewise-smooth curve C.

Notations And Definitions:

DEFINITION 7. *A piecewise-smooth curve is called a* **path**.

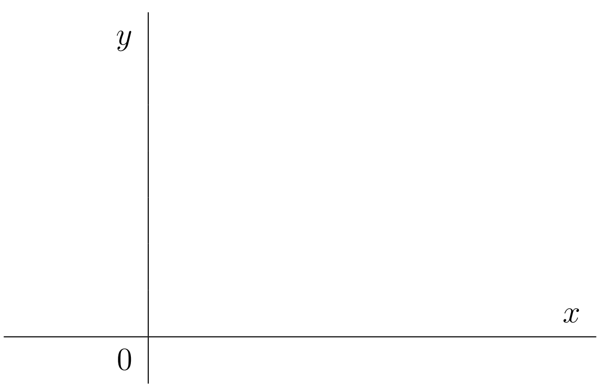
- **Types of curves:**

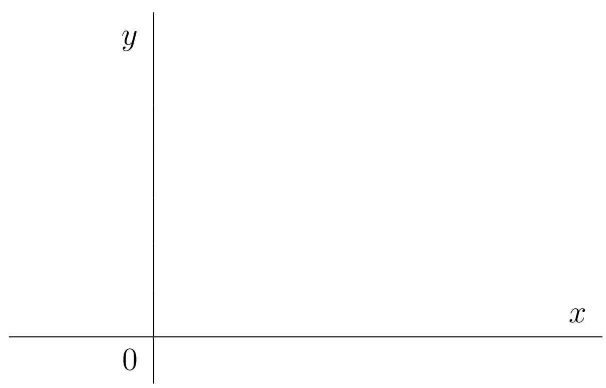
simple not closed	not simple not closed	simple closed	not simple, closed

- **Types of regions:**

simply connected	not simply connected

- **Convention:** The **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. If C is given by $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}, a \leq t \leq b$, then the region D bounded by C is always on the left as the point $\mathbf{r}(t)$ traverses C.





- The positively oriented boundary curve of D is denoted by ∂D.

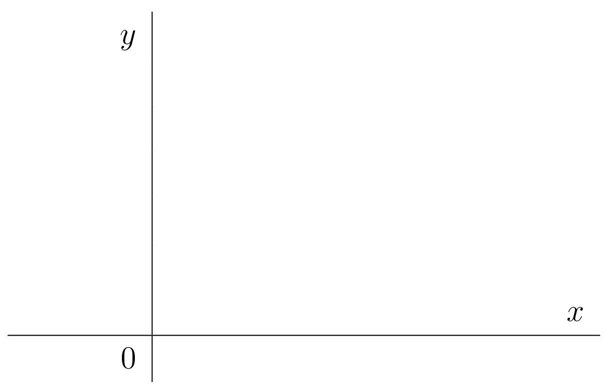
•**GREEN's THEOREM**: *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region that contains D, then*

$$\oint_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A.$$

EXAMPLE 8. *Evaluate:*

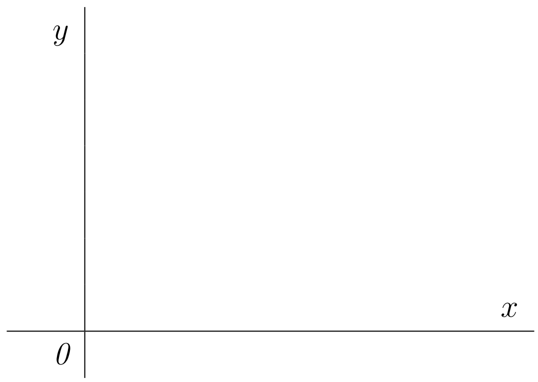
$$I = \oint_C e^x(1-\cos y)\,\mathrm{d}x - e^x(1-\sin y)\,\mathrm{d}y$$

where C is the boundary of the domain $D = \{(x,y) : 0 \le x \le \pi, 0 \le y \le \sin x\}$.



EXAMPLE 9. *Let C be a triangular curve consisting of the line segments from $(0,0)$ to (5,0), from $(5,0)$ to $(0,5)$, and from $(0,5)$ to $(0,0)$. Evaluate the following integrals:*

(a) $I_1 = \oint_C (x^2y + \frac{1}{2}y^2)\,\mathrm{d}x + (xy + \frac{1}{3}x^3 + 3x)\,\mathrm{d}y$



(b) $I_2 = \oint_C (x^2y + \frac{1}{2}y^2 + e^{x\sin x})\,\mathrm{d}x + (xy + \frac{1}{3}x^3 + x - 4\arctan(e^y))\,\mathrm{d}y$

(c) $I_3 = \oint_C (x^2y + \frac{1}{2}y^2 - 55\arcsin(\sec x))\,\mathrm{d}x + (12y^5\cos y^3 + xy + \frac{1}{3}x^3 + x)\,\mathrm{d}y$

• **Application: Computing areas.**

$$A(D) = \oint_{\partial D} x\,\mathrm{d}y = -\oint_{\partial D} y\,\mathrm{d}x = \frac{1}{2}\oint_{\partial D} x\,\mathrm{d}y - y\,\mathrm{d}x.$$

EXAMPLE 10. *Find the area enclosed by the ellipse* $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

SUMMARY: Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ *be a vector field on an open simply connected domain* D*. Suppose that* P *and* Q *have continuous partial derivatives through* D*. Then the facts below are equivalent.*

The field $\mathbf{F}$ is **conservative**on D	$\Longleftrightarrow$	There exists f s.t. $\nabla f = \mathbf{F}$

The field $\mathbf{F}$ is **conservative** on D	$\Longleftrightarrow$	$\int_{\breve{AB}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D

The field $\mathbf{F}$ is **conservative** on D	$\Longleftrightarrow$	$\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ throughout D

The field $\mathbf{F}$ is **conservative** on D	$\Longleftrightarrow$	$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C in D

EXAMPLE 11. *Determine whether or not the vector field is conservative:*

(a) $\mathbf{F}(x, y) = \langle x^2 + y^2, 2xy \rangle$.

(b) $\mathbf{F}(x, y) = \langle x^2 + 3y^2 + 2, 3x + ye^y \rangle$

EXAMPLE 12. *Given* $\mathbf{F}(x, y) = \sin y\mathbf{i} + (x \cos y + \sin y)\mathbf{j}$.

(a) *Show that* $\mathbf{F}$ *is conservative.*

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$

(c) *Find the work done by the force field* $\mathbf{F}$ *in moving a particle from the point* $(3, 0)$ *to the point* $(0, \pi/2)$.

(d) *Evaluate* $\oint_C \mathbf{F}\, \mathrm{d}\mathbf{r}$ *where* C *is an arbitrary path in* $\mathbb{R}^2$.

EXAMPLE 13. *Given*

$$\mathbf{F} = \left\langle 2xy^3 + z^2, 3x^2y^2 + 2yz, y^2 + 2xz \right\rangle.$$

Find a function f *s.t.* $\nabla f = \mathbf{F}$