

14.8: STOKES' THEOREM

Stokes' Theorem can be regarded as a 3-dimensional version of Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D \text{curl} \mathbf{F} \cdot \mathbf{k} dA.$$

Let S be an oriented surface with unit normal vector $\hat{\mathbf{n}}$ and with the boundary curve C (which is a space curve).

The orientation on S induces the **positive orientation of the boundary curve** C : if you walk in the positive direction around C with your head pointing in the direction of $\hat{\mathbf{n}}$, then the surface will always be on your left.

The positively oriented boundary curve of an oriented surface S is often written as ∂S .

Stokes' Theorem: *Let S be an oriented piece-wise-smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S},$$

or

$$\iint_S \text{curl} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

EXAMPLE 1. Find the work performed by the forced field $\mathbf{F}(x, y, z) = \langle 3x^8, 4xy^3, y^2x \rangle$ on a particle that traverses the curve C in the plane $z = y$ consisting of 4 line segments from $(0, 0, 0)$ to $(1, 0, 0)$, from $(1, 0, 0)$ to $(1, 3, 3)$, from $(1, 3, 3)$ to $(0, 3, 3)$, and from $(0, 3, 3)$ to $(0, 0, 0)$.

EXAMPLE 2. Verify Stokes' Theorem $\iint_S \text{curl} \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$ for the vector field $\vec{F} = \langle 3y, 4z, -6x \rangle$ and the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = -7$ and oriented upward. Be sure to check and explain the orientations.

Solution: Use the following steps:

- Parametrize the boundary circle ∂S and compute the line integral.

- Parametrize the surface of the paraboloid and compute the surface integral:

THEOREM 3. *If \mathbf{F} is a vector field defined on \mathbb{R}^3 whose component functions have continuous partial derivatives and $\text{curl}\mathbf{F} = 0$, then \mathbf{F} is a conservative vector field.*

SUMMARY: *Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a continuous vector field in \mathbb{R}^3 .*

There exists f s.t.
 $\nabla f = \mathbf{F}$

$\int_{\widetilde{AB}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path

\mathbf{F} is conservative
in \mathbb{R}^3

$\text{curl}\mathbf{F} = \mathbf{0}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C