5 FUNCTIONS

5.1 Definition and Basic Properties

DEFINITION 1. Let $X$ and $Y$ be nonempty sets. A function $f$ from the set $X$ to the set $Y$ is a correspondence that assigns to each element $x$ in the set $X$ one and only one element $y$ in the set $Y$, which is denoted by $f(x)$.

We call $X$ the domain of $f$ and $Y$ the codomain of $f$.

If $x \in X$ and $y \in Y$ are such that $y = f(x)$, then $y$ is called the value of $f$ at $x$, or the image of $x$ under $f$. We may also say that $f$ maps $x$ to $y$.

Using diagram

DEFINITION 2. Two functions $f$ and $g$ are equal if they have the same domain and the same codomain and if $f(x) = g(x)$ for all $x$ in domain.

DEFINITION 3. The graph of $f : X \to Y$ is the set

$$G_f = \{(x, y) \in X \times Y | y = f(x)\}.$$ 

- We can determine a function from its domain, codomain, and graph.
- We can describe a function by formula, by listing its values, or by words.

EXAMPLE 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 3$, $g : \mathbb{R} \to [0, \infty)$ be defined by $g(x) = x^2 + 3$, and $h : \{-1, 0, 1\} \to \mathbb{R}$ be defined by $h(x) = x^2 + 3$.

(a) Determine whether $f = g$.

(b) Determine whether $f = h$.

(c) Find the graphs of $f$, $g$, and $h$. 
EXAMPLE 5. Decide if the following diagrams define functions from $A$ to $B$.

EXAMPLE 6. Let $X = \{2, 4, 6\}$ and $Y = \{a, b, c, d\}$. Let $f$ be a function defined by $f(2) = b, f(4) = a, f(6) = d$ and let $g$ be a function from $X$ to $Y$ defined by its graph $G_g = \{(2, c), (4, c), (6, c)\}$ Find the following.

(a) The image of 2 under $f$.
(b) The image of 6 under $g$.
(c) The preimage of $d$ under $f$.
(d) The preimage of $c$ under $g$.
(e) The preimage of $d$ under $g$.
(f) The codomain of $g$.
(g) $G_f$

Range (or Image) of a Function

DEFINITION 7. Let $f : X \rightarrow Y$ be a function. The range of $f$ (also called the image of $f$) is the set

$$\{y \in Y | y = f(x) \text{ for some } x \in X\}.$$ 

We denote the range (or image) of the function $f$ by $\text{ran}f$ (or $\text{Im}f$).

EXAMPLE 8. Let $f : X \rightarrow Y$ be a function. Using symbols complete the following

- $\text{ran}f \subseteq \_\_\_$
- $\forall y \in Y, y \in \text{ran}f \iff \_\_\_$
- $y \notin \text{ran}f \iff \_\_\_$
EXAMPLE 9. $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = \cos x$. Find $\text{ran } f$.

EXAMPLE 10. Let $f : [\frac{1}{3}, \infty) \rightarrow \mathbb{R}$ be defined by $f(x) = \sqrt{3x - 1}$ and $S = \{y \in \mathbb{R} | y \geq 0\}$. Prove that $\text{ran } f = S$.

5.2 Composition of Functions

DEFINITION 11. Let $A$, $B$, and $C$ be nonempty sets, and let $f : A \rightarrow B$, $g : B \rightarrow C$. We define a function

$$g \circ f : A \rightarrow C,$$

called the composition of $f$ and $g$, by

$$(g \circ f)(a) =$$

Using diagram
EXAMPLE 12. Let \( A = \{1, 2, 3, 4\} \), \( B = \{a, b, c, d\} \), \( C = \{r, s, t, u, v\} \) and define the functions \( f : A \to B \), \( g : B \to C \) by their graphs:

\[
G_f = \{(1, b), (2, d), (3, a), (4, a)\}, \quad G_g = \{(a, u), (b, r), (c, r), (d, s)\}.
\]

Find \( g \circ f \). What about \( f \circ g \)?

EXAMPLE 13. Let \( f, g : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = e^x \) and \( g(x) = x \sin x \). Find \( f \circ g \) and \( g \circ f \).
PROPOSITION 14. Let $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then

$$(h \circ g) \circ f = h \circ (g \circ f),$$

i.e. composition of functions is associative.

Proof.

Section 5.3 Surjective (or onto) and Injective (or one-to-one) Functions

Surjective functions (“onto”)

DEFINITION 15. Let $f : X \to Y$ be a function. Then $f$ is surjective (or a surjection) if the range of $f$ coincides with its codomain, i.e.

$$\text{ran} f = Y.$$

Note: surjection is also called “onto”.

Proving surjection:

We know that for all $f : X \to Y$:  

Thus, to show that $f : X \to Y$ is a surjection it is sufficient to prove that  

In other words,

**to prove that $f : X \to Y$ is a surjective function it is sufficient to show that**

Question: How to disprove surjectivity?
EXAMPLE 16. Let \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to [0, \infty) \) defined by \( f(x) = g(x) = x^4 \). Determine whether the following are true

(a) \( \text{ran} \, f = \text{ran} \, g \)

(b) \( f = g \)

(c) \( f \) is surjective

(d) \( g \) is surjective

EXAMPLE 17. Prove that the function \( f : \mathbb{R} - \{2\} \to \mathbb{R} - \{3\} \) defined by \( f(x) = \frac{3x}{x - 2} \) is surjective.
Injective functions (“one to one”)

DEFINITION 18. Let \( f : X \to Y \) be a function. Then \( f \) is injective (or an injection) if whenever \( x_1, x_2 \in X \) and \( x_1 \neq x_2 \), we have \( f(x_1) \neq f(x_2) \).

In other words, \( f \) is injective if and only if the ranges of every two distinct points in the domain of \( f \) are distinct.

EXAMPLE 19. Given \( X = \{1, 2, 3\} \) and \( Y = \{3, 4, 5\} \). Determine whether the following functions are injective. Justify your answer.

(a) \( f : X \to Y \) defined by \( G_f = \{(1, 3), (2, 4), (3, 5)\} \)

(b) \( g : X \to Y \) defined by \( G_g = \{(1, 5), (2, 4), (3, 4)\} \)

Proving injection:

Let \( P(x_1, x_2) : x_1 \neq x_2 \) and \( Q(x_1, x_2) : f(x_1) \neq f(x_2) \).

Then by definition \( f \) is injective if ________________.

Using contrapositive, we have ________________.

In other words, to prove injection show that:

Question: How to disprove injectivity?

EXAMPLE 20. Prove or disprove injectivity of the following functions.

(a) \( f : \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{x} \).
(b) \( f : \mathbb{R} \to \mathbb{R}, f(x) = x^4. \)

(c) \( f : \mathbb{Z} \to \mathbb{Z}, f(n) = \begin{cases} 
    n/2 & \text{if } n \in \mathbb{E}, \\
    2n & \text{if } n \in \mathbb{O}.
\end{cases} \)

(d) \( f : \mathbb{Z} \to \mathbb{Z}, f(n) = \begin{cases} 
    n & \text{if } n \in \mathbb{E}, \\
    5n & \text{if } n \in \mathbb{O}.
\end{cases} \)

Discussion Exercise.

- Must a strictly increasing or decreasing function be injective?
• Must an injective function be strictly increasing or decreasing?

EXAMPLE 21. Prove or disprove injectivity of the following functions. In each case, \( f : \mathbb{R} \to \mathbb{R} \).

(a) \( f(x) = 3x^5 + 5x^3 + 2x + \pi \).

(b) \( f(x) = x^{12} + x^8 - x^4 + 12 \).

Bijective functions

DEFINITION 22. A function that is both surjective and injective is called **bijective** (or bijection.)

\( f \) is not bijective \( \iff \)
PROPOSITION 23. A function $f$ is bijective if and only if every point in $\text{codom } f$ has a unique preimage in the $\text{dom } f$.

EXAMPLE 24. Determine which of the following functions are bijective.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$.  
(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

PROPOSITION 25. Let $f : A \to B$ and $g : B \to C$. Then

i. If $f$ and $g$ are surjections, then $g \circ f$ is also a surjection.

Proof.

ii. If $f$ and $g$ are injections, then $g \circ f$ is also an injection.

Proof.

COROLLARY 26. If $f$ and $g$ are bijections, then $g \circ f$ is also a bijection.
Identity Function

For a set $X$ we define the identity function $I_X : X \rightarrow X$ by the rule $I_X(x) = x$ for all $x \in X$. In other words, the identity function maps every element to itself.

Though this seems like a rather trivial concept, it is useful and important.

**Proposition 27.** Let $f : X \rightarrow Y$. Then $f \circ I_X = f$ and $I_Y \circ f = f$.

### 5.4 Invertible Functions

**Inverse Functions**

**Definition 28.** Let $f : X \rightarrow Y$ be a function. We say that $f$ is invertible if there is a function $g : Y \rightarrow X$ such that for all $x \in X$ and for all $y \in Y$,

$$y = f(x) \iff x = g(y).$$

We say that such a function $g$ is an inverse function of $f$.

**Question 1** What is the inverse of $g$?

**Question 2** Are the functions in Example 6 invertible?

**Remark 29.** $f$ is invertible if and only if its inverse is invertible.

**Example 30.** Show that the function $f : \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{3\}$ defined by $f(x) = \frac{3x}{x - 2}$ is invertible and find its inverse function. (Note that the given function is bijective.)
PROPOSITION 31. A function $f : X \to Y$ is invertible if and only if there exists a function $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$.

PROPOSITION 32. The inverse function is unique.

Proof.

Notation

When $f : X \to Y$ is invertible, the unique inverse function is denoted by $f^{-1}$, and $f^{-1} : Y \to X$. 
REMARK 33. Finding the inverse of a bijective function is not always possible by algebraic manipulations. For example,

\[ \text{if } f(x) = e^x \text{ then } f^{-1}(x) = \quad \]

The function \( f(x) = 3x^5 + 5x^3 + 2x + 220 \) is known to be bijective, but there is no way to find expression for its inverse.

THEOREM 34. A function \( f : X \to Y \) is invertible if and only if \( f \) is bijective.

COROLLARY 35. If a function \( f : X \to Y \) is bijective, so is \( f^{-1} \).