14: Nonhomogeneous Equations. Method of Undetermined Coefficients (section 3.5)

1. If $y_1(t)$ and $y_2(t)$ are two solutions of a second-order *nonhomogeneous* linear ODE,

$$y'' + p(t)y' + q(t)y = g(t), \quad g(t) \not\equiv 0.$$

Then $y_1(t) - y_2(t)$ is a particular solution of the corresponding homogeneous equation,

$$y'' + p(t)y' + q(t)y = 0.$$

2. *THEOREM: Solution of Nonhomogeneous Linear Equation*

Let

$$y'' + p(t)y' + q(t)y = g(t), \quad g(t) \not\equiv 0$$

be a second-order *nonhomogeneous* linear differential equation. If $y_p(t)$ is a particular solution of this equation and $y_h(t)$ is the general solution of the corresponding homogeneous equation,

$$y'' + p(t)y' + q(t)y = 0,$$

then

$$y(t) = y_h(t) + y_p(t)$$

is the general solution of the nonhomogeneous equation.

3. To solve a nonhomogeneous linear ODE:

Step 1: Find a particular solution of a nonhomogeneous linear ODE.

Step 2: Find general solution of the corresponding homogeneous linear ODE.

Step 3: Add the results of Steps 1&2.

4. One solution of $y'' - y = t$ is $y(t) = -t$, as you can verify. What is the general solution?

5. Consider

$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) \tag{1}$$

. If $y(t) = y_p(t)$ and $y(t) = Y_p(t)$ are particular solutions of

$$y'' + p(t)y' + q(t)y = g_1(t)$$

and

$$y'' + p(t)y' + q(t)y = g_2(t),$$

respectively, then $y(t) = y_p(t) + Y_p(t)$ is a particular solution of (1).

Method of Undetermined Coefficients

6. Consider a particular class of nonhomogeneous linear ODE with constant coefficients

$$ay'' + by' + cy = g(t),$$

where a, b, c are real constants and $g(t)$ involves linear combinations, sums and products of

$$t^m, \quad e^{\alpha t}, \quad \sin(\beta t), \quad \cos(\beta t).$$

7. The idea of the Method of Undetermined Coefficients: guess a particular solution $y_p(t)$ using a generalized form of $g(t)$. Then by substitution determine the coefficients for the generalized solution.

8. Find general solution:

   (a) $y'' - 3y' + 2y = 4e^{3t}$

   (b) $y'' - 3y' + 2y = 4e^{t}$

   (c) $y'' + 10y' + 25y = 3e^{-5t}$

9. **Multiplicity s of a given number $\alpha + i\beta$**

   If $g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n)e^{\alpha t}, \quad \beta = 0$, then

   - α doesn't coincide with a root of characteristic polynomial $\Rightarrow s = 0$.
   - α coincides with a non repeated root of characteristic polynomial $\Rightarrow s = 1$.
   - α coincides with a repeated root of characteristic polynomial $\Rightarrow s = 2$.

   If $g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_1 t^{n-1} + \ldots + B_n)e^{\alpha t}\cos(\beta t)$ or $g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n)e^{\alpha t}\sin(\beta t)), \quad \beta \neq 0$, then

   - $\alpha + i\beta$ doesn't coincide with a root of characteristic polynomial $\Rightarrow s = 0$.
   - $\alpha + i\beta$ coincides with a (complex) root of characteristic polynomial $\Rightarrow s = 1$.

10. If $g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n)e^{\alpha t}\cos(\beta t)$ or $g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n)e^{\alpha t}\sin(\beta t))$ then we choose a particular solution in the form

$$y_p(t) = t^s(A_0 t^n + A_1 t^{n-1} + \ldots + A_n)e^{\alpha t}\cos(\beta t) + t^s(D_0 t^n + D_1 t^{n-1} + \ldots + D_n)e^{\alpha t}\sin(\beta t),$$

respectively.

11. Find general solution of

$$y'' + 2y' + 5y = 3\sin 2t$$

12. Determine a suitable choice for y_p:

	$ay'' + by' + cy = g(t)$	r_1, r_2	$\alpha + i\beta$	s	$y_p(t)$
1	$y'' = e^t$				
2	$y'' - 4y' + 4 = e^{2t}$				
3	$y'' + 2y' + 10y = 6\sin(3t)$	$r_{1,2} = -1 \pm 3i$			
4	$y'' + 2y' + 10y = 6e^{-t}\sin(3t)$	$r_{1,2} = -1 \pm 3i$			
5	$y'' + 2y' + 10y = (t^3 - 1)\sin(3t)$	$r_{1,2} = -1 \pm 3i$			
6	$y'' + 2y' + 10y = t^2e^{-t}\sin(3t)$	$r_{1,2} = -1 \pm 3i$			
7	$y'' = 2t - 2013$	$r_1 = r_2 = 0$			