15: Method Variation of Parameters (MVP) (section 3.6)

1. Preliminary: For the first order linear nonhomogeneous equation $y' + p(t)y = g(t)$, the MVP is the alternative to the Method of Integrating Factor. But MVP is more conceptual and can be generalized to the higher order ODE.

   The idea: if $y_1(t)$ is a non-zero solution of the corresponding homogeneous equation, then $y(t) = Cy_1(t)$ is general solution of the corresponding homogeneous equation. Variate the parameter (constant) C and seek a solution of the nonhomogeneous ODE in the form
   $y(t) = u(t)y_1(t)$.

2. Consider a second-order *nonhomogeneous* linear DE

$$y'' + p(t)y' + q(t)y = g(t).$$

   If $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions of the corresponding homogeneous equation, then the general solution of the corresponding homogeneous equation will be $y(t) = C_1y_1(t) + C_2y_2(t)$. Variate the parameters C_1, C_2 and seek a solution of the nonhomogeneous ODE in the form

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

   Then Apply Product Rule and Cramer Rule to get

$$u_1' = -\frac{g(t)y_2(t)}{W(y_1, y_2)(t)}, \quad u_2' = \frac{g(t)y_1(t)}{W(y_1, y_2)(t)}$$

3. Advantage of MVP over the Method of undetermined coefficients:

   - MVP always will yield a particular solution provided the associated homogeneous ODE can be solved.
   - MVP is not limited to a function $g(x)$ that is a combination of $t^m, e^{\alpha t}, \sin(\beta t), \cos(\beta t)$.

4. Solve $y'' + y = \tan t, \quad 0 < t < \pi/2$.