17: Forced Vibrations (section 3.8)

1. Suppose now we take into consideration an **external force** $F(t)$ acting on a vibrating spring/mass system. The inclusion of $F(t)$ in the formulation of Newton's second law yields
$$mu'' = mg - k(L+u) - \gamma u' + F(t),$$
or taking into account that $mg = kL$ we get the so called DE of **forced motion**
$$mu'' + \gamma u' + ku = F(t). \tag{1}$$
For this DE we have the same initial conditions as for unforced vibration. Namely,
$$u(0) = u_0, \quad u'(0) = v_0,$$
where u_0 is the initial displacement and v_0 is the initial velocity.

Case of Periodic External Force

2. When $F(t)$ is a periodic function such as
$$F(t) = F_0 \sin(\omega t) \quad \text{or} \quad F(t) = F_0 \cos(\omega t)$$
then by the Method of Undetermined coefficients a particular solution of forced motion can be obtained as
$$u_p(t) = t^s(A\cos(\omega t) + B\sin(\omega t)) = t^s R\cos(\omega t - \delta)$$
Remind that $R = \sqrt{A^2 + B^2}$ is amplitude and δ is phase ($\cos\delta = A/R, \sin\delta = B/R$)

3. Characteristic equation $mr^2 + \gamma r + k = 0$. We consider the case
$$D = \gamma^2 - 4mk < 0,$$
i.e.
$$\gamma < 2\sqrt{km} =: \gamma_{crit}$$
because otherwise there are no unforced oscillations as it follows from the section 3.7.

4. Recall that the roots of the characteristic equation when $D < 0$ are
$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{-\gamma \pm i\sqrt{4mk - \gamma^2}}{2m} = -\frac{\gamma}{2m} \pm i\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} = \lambda \pm i\sqrt{\omega_0^2 - \lambda^2} = \lambda \pm i\mu$$
where $\omega_0^2 = \frac{k}{m}$.

5. Solution of the corresponding homogeneous equation is
$$u_h(t) = e^{\lambda t}(C_1\cos(\mu t) + C_2\sin(\mu t)) = R_1 e^{\lambda t}\cos(\mu t - \delta),$$
where $R_1 = \sqrt{C_1^2 + C_2^2}$ and $\cos\delta = C_1/R_1, \sin\delta = C_2/R_1$.

6. The general solution of (1) is $u(t) = u_p(t) + u_h(t)$, or
$$u(t) = t^s R\cos(\omega t - \delta) + R_1 e^{\lambda t}\cos(\mu t - \delta). \tag{2}$$

Forced Damped Vibration

Steady-State and Transient solutions

7. Motion with damping means $\gamma \neq 0$. It implies the following

   - γ is a positive constant and then $\lambda = -\frac{\gamma}{2m} < 0$.
   - $r_{1,2} \neq \omega$, hence, $s = 0$.
   - The general solution of (1) in this case will be

$$u(t) = \underbrace{R\cos(\omega t - \delta)}_{\text{steady-state solution}} + \underbrace{R_1 e^{\lambda t}\cos(\mu t - \delta)}_{\text{transient solution}, u_c(t)}. \tag{3}$$

   Emphasize, that transient solution $u_c(t)$ dies of as time increases (because $\lambda < 0$), i.e.

$$\lim_{t\to\infty} u_c(t) = 0.$$

   Thus, for large values of t, the displacements of mass are closely approximated by $u_p(t)$:

$$u(t) \approx R\cos(\omega t - \delta).$$

Forced UnDamped Vibration

8. Motion without damping means $\gamma = 0$ and then we have the following IVP:

$$mu'' + ku = F(t), \quad u(0) = u_0, \quad u'(0) = v_0,$$

or

$$u'' + \omega_0^2 u = \frac{F(t)}{m}, \quad u(0) = u_0, \quad u'(0) = v_0.$$

9. Consider the particular case of a periodic external force:

$$u'' + \omega_0^2 u = F_0\cos(\omega t).$$

10. In this case $\lambda = -\frac{\gamma}{2m} = 0, \quad \mu = \sqrt{\omega_0^2 - \lambda^2} = \omega_0$. Thus, general solution is

$$u(t) = Rt^s\cos(\omega t - \delta) + R_1\cos(\omega_0 t - \delta).$$

   - Case 1: $\omega = \omega_0$, i.e. $s = 1$.

11. This is the case when the frequency of the external force coincides with the natural frequency of the system.

12. General solution in this case

$$u(t) = Rt\cos(\omega t - \delta) + R_1 \cos(\omega_0 t - \delta)$$

It follows that $u(t)$ is unbounded as time increases (This phenomenon is known as **pure resonance**.)

- Case 2: $\omega \neq \omega_0$, i.e. $s = 0$.

13. This is the case when the frequency of the external force does not coincide with the natural frequency of the system.

14. Consider the following particular case (when mass is initially at rest):

$$u'' + \omega_0^2 u = F_0 \cos(\omega t), \quad u(0) = 0, \quad u'(0) = 0.$$

15. One can show (see your homework) that the general solution of this ODE is

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left(\cos(\omega t) - \cos(\omega_0 t)\right).$$

Using trigonometric identity

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$$

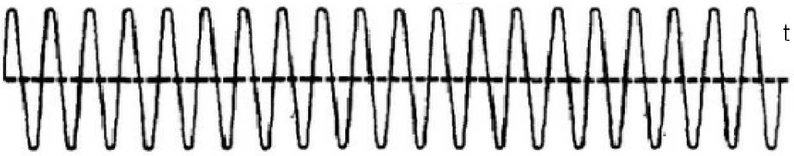
we rewrite the general solution as

$$u(t) = \underbrace{\left[-\frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\frac{\omega-\omega_0}{2}t\right)\right]}_{\text{slowly varying sinusoidal amplitude}} \sin\left(\frac{\omega+\omega_0}{2}t\right)$$

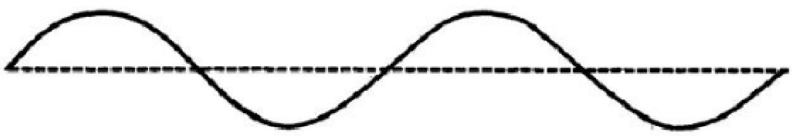
If $|\omega - \omega_0|$ is small, then $\omega + \omega_0$ is much greater than $|\omega - \omega_0|$. Consequently, $\sin\left(\frac{\omega+\omega_0}{2}t\right)$ is rapidly oscillation function comparing to $\sin\left(\frac{\omega-\omega_0}{2}t\right)$ with a slowly varying sinusoidal amplitude $\frac{2F_0}{m|\omega_0^2-\omega^2|}\sin\left(\frac{\omega-\omega_0}{2}t\right)$.

16. Type of motion processing *a periodic variation of amplitude* is called an amplitude modulation effect (AM).

carrier $\sin\left(\frac{\omega+\omega_0}{2}t\right)$



signal $-\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\left(\frac{\omega-\omega_0}{2}t\right)$



amplitude modulated wave $u(t)=\left[-\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\left(\frac{\omega-\omega_0}{2}t\right)\right]\sin\left(\frac{\omega+\omega_0}{2}t\right)$