

## 20: Step functions. Differential equations with discontinuous forcing functions (sections 6.3 and 6.4)

1. Consider the  $n$ -th order linear ODE with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_1 y' + a_0 y = g(t),$$

where  $g(t)$  is a piecewise continuous function (function with jump discontinuities).

Jump discontinuities occur naturally in engineering problems such as electrical circuits with on/off switches. To handle such behavior, Heaviside introduced the following step function.

2. **Unit Step Function**  $u_c(t)$  ( $c \geq 0$ ) is defined by

$$u_c(t) = \begin{cases} 0, & 0 \leq t < c \\ 1, & t \geq c \end{cases}$$

3. When a function  $f(t)$  defined for  $t \geq 0$  is multiplied by  $u_c(t)$ , this unit step function "turns off" a portion of the graph of that function. For example, consider  $(t^2 + 1)u_3(t)$ .

4. *FACT 1.* Any function with jump discontinuities at  $t = c_1, c_2, \dots, c_k$  can be represented in terms of unit step functions. In other words, we can use unit step function to write a piecewise-defined functions in a compact form.

5. Express  $f$  in terms of unit step function

$$(a) f(t) = \begin{cases} 4, & 0 \leq t < 3 \\ 1, & 3 \leq t < 5 \\ -2, & 5 \leq t \end{cases}$$

$$(b) f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & a \leq t \end{cases}$$

$$(c) f(t) = \begin{cases} 3, & 0 \leq t < 2 \\ 1, & 2 \leq t < 3 \\ t, & 3 \leq t < 5 \\ t^2, & 5 \leq t \end{cases}$$

6. **FACT 2. Translation in  $t$  property** for Laplace Transform: if  $F(s) = \mathcal{L}\{f(t)\}$  then

$$F(s) = \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s).$$

7. Find  $\mathcal{L}\{u_c(t)\}$

8. Duality between Laplace transform and its inverse:

Derivative	$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	$\mathcal{L}^{-1}\{F'(s)\} = -tf(t)$
Translation	$\mathcal{L}\{e^{\alpha t}f(t)\} = F(s - \alpha)$	$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t - c)$

9. Let  $f(t)$  will be the same as in 5(c). Find  $\mathcal{L}\{f\}$ .

10. Find the inverse Laplace transform of

$$H(s) = \frac{e^{-4s}}{s^2 + 9} + \frac{se^{-3s}}{s^2 + 4}$$

11. Let

$$g(t) = \begin{cases} 20, & 0 \leq t < 3\pi, \\ 0, & 3\pi \leq t < 4\pi \\ 20, & 4\pi \leq t \end{cases}$$

(a) Solve IVP:

$$y'' + 2y' + 2y = g(t), \quad y(0) = 10, \quad y'(0) = 0,$$

Solution:

**Step 1.** Express  $g(t)$  in compact form.

**Step 2.** Find  $\mathcal{L}\{g(t)\} = G(s)$ .

**Step 3.** Find  $\mathcal{L}\{y'' + 2y' + 2y\}$ .

**Step 4.** Combine steps 2& 3 to get  $\mathcal{L}\{y(t)\} = Y(s)$ .

**Step 5.** Apply inverse Laplace transform to find  $y(t)$ . This step usually requires partial fraction decomposition.

(b) Sketch the graph of  $y(t)$ .