20: Impulse Function (section 6.5)

1. In applications (mechanical systems, electrical circuits etc) one encounters functions (external force) of large magnitude that acts only for a very short period of time. To deal with violent forces of short duration the so called delta function is used. This function was introduces by Paul Dirac.

2. If a force $F(t)$ acts on a body of mass m on the time interval $[t_0, t_1]$, then the impulse due to F is defined by the integral

$$\text{impulse} = \int_{t_0}^{t_1} F(t)\mathrm{d}t = \int_{t_0}^{t_1} ma(t)\mathrm{d}t = \int_{t_0}^{t_1} m\frac{\mathrm{d}v(t)}{\mathrm{d}t}\mathrm{d}t = mv(t_1) - mv(t_0)$$

3. The impulse equals the change in momentum.

   When a hammer strikes an object, it transfers momentum to the object. This change in momentum takes place over a very short period of time. The change in momentum (=the impulse) is the area under the curve defined by $F(t)$

$$\text{the total impulse of the force } F(t) = \int_{-\infty}^{\infty} F(t)\mathrm{d}t$$

4. Consider a family of piecewise functions (forces)

$$d_\tau = \begin{cases} \dfrac{1}{2\tau}, & \text{if} \quad |t| < \tau \\ 0, & \text{if} \quad |t| \geq \tau. \end{cases}$$

   Then all forces d_τ have the total impulse which is equal 1.

5. Dirac DELTA Function: In practice it is convenient to work with another type of unit impulse, an idealized unit impulse force that concentrated at $t = 0$:

$$\lim_{\tau \to 0} d_\tau(t) = \delta(t).$$

6. Definition The Dirac Delta Function, $\delta(t)$, is characterized by the following 2 properties:

   (a) $\delta(t) = 0$ for all $t \neq 0$.

   (b) $\int_{-\infty}^{\infty} \delta(t)f(t)\mathrm{d}t = f(0)$ for any f$f(t)$ continuous on an open interval containing $t = 0$.

   Note that δ-function does not behave like an ordinary function.

7. A unit impulse concentrated at $t = t_0$ is denoted by $\delta(t - t_0)$ and

$$\int_{-\infty}^{\infty} \delta(t - t_0)f(t)\mathrm{d}t = f(t_0), \quad t \neq t_0.$$

8. Laplace Transform of *delta*-function:

$$\mathcal{L}\{\delta(t)\} = 1$$

For $t_0 \geq 0$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

9. Solve the given IVP and sketch the graph of the solution:

$$y'' + y = \delta(t-2\pi), \quad y(0)=0, \quad y'(0)=1.$$

10. Solve $2y'' + y' + 4y = \delta(t - \frac{\pi}{6})\sin t$ subject to $y(0)=0,\ y'(0)=0$.

11. Remark:

$$\int_{-\infty}^{t} \delta(t-t_0)\mathrm{d}t = \begin{cases} 0, & t < t_0 \\ 1, & t \geq t_0 \end{cases} = u_{t_0}(t).$$

In other words, derivative of unit step function is *delta*-function.

21: Convolution Integral (section 6.6)

1. If f and g are piecewise continuous on $[0,\infty)$, then the **convolution**, $f * g$, is defined by the integral

$$f * g = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau.$$

2. Convolution is commutative, i.e. $f * g = g * f$

3. Convolution Theorem. If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{G(t)\}$ exist for $s \geq a > 0$ then for $s > a$

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s),$$

or

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g.$$

4. Use the convolution integral to compute

   (a) $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}$

   (b) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

5. Consider IVP:

$$y'' + \omega^2 y = g(t), \quad y(0)=0, \quad y'(0)=1.$$

   (a) Express the solution of the given IVP in terms of the convolution integral.

   (b) Use the Methof of Variation of Parameters to solve the given IVP and compare the result with (a).