

20: Impulse Function (section 6.5)

1. In applications (mechanical systems, electrical circuits etc) one encounters functions (external force) of large magnitude that acts only for a very short period of time. To deal with violent forces of short duration the so called **delta function** is used. This function was introduced by Paul Dirac.
2. If a force $F(t)$ acts on a body of mass m on the time interval $[t_0, t_1]$, then the **impulse** due to F is defined by the integral

$$\text{impulse} = \int_{t_0}^{t_1} F(t)dt = \int_{t_0}^{t_1} ma(t)dt = \int_{t_0}^{t_1} m \frac{dv(t)}{dt} dt = mv(t_1) - mv(t_0)$$

3. The impulse equals the change in momentum.

When a hammer strikes an object, it transfers momentum to the object. This change in momentum takes place over a very short period of time. The change in momentum (=the impulse) is the area under the curve defined by $F(t)$

$$\text{the total impulse of the force } F(t) = \int_{-\infty}^{\infty} F(t)dt$$

4. Consider a family of piecewise functions (forces)

$$d_\tau = \begin{cases} \frac{1}{2\tau}, & \text{if } |t| < \tau \\ 0, & \text{if } |t| \geq \tau. \end{cases}$$

Then all forces d_τ have the total impulse which is equal 1.

5. Dirac DELTA Function: In practice it is convenient to work with another type of unit impulse, an idealized unit impulse force that concentrated at $t = 0$:

$$\lim_{\tau \rightarrow 0} d_\tau(t) = \delta(t).$$

6. Definition The Dirac Delta Function, $\delta(t)$, is characterized by the following 2 properties:

(a) $\delta(t) = 0$ for all $t \neq 0$.

(b) $\int_{-\infty}^{\infty} \delta(t)f(t)dt = f(0)$ for any $f(t)$ continuous on an open interval containing $t = 0$.

Note that δ -function does not behave like an ordinary function.

7. A unit impulse concentrated at $t = t_0$ is denoted by $\delta(t - t_0)$ and

$$\int_{-\infty}^{\infty} \delta(t - t_0)f(t)dt = f(t_0), \quad t \neq t_0.$$

8. Laplace Transform of *delta*-function: $\mathcal{L}\{\delta(t)\} =$ For $t_0 \geq 0$

$$\mathcal{L}\{\delta(t - t_0)\} =$$

9. Solve the given IVP and sketch the graph of the solution:

$$y'' + y = \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 1.$$

10. Solve $2y'' + y' + 4y = \delta(t - \frac{\pi}{6}) \sin t$ subject to $y(0) = 0$, $y'(0) = 0$.

11. Remark:

$$\int_{-\infty}^t \delta(t - t_0) dt = \begin{cases} 0, & t < t_0 \\ 1, & t \geq t_0 \end{cases} = u_{t_0}(t).$$

In other words, derivative of unit step function is *delta*-function.

21: Convolution Integral (section 6.6)

1. If f and g are piecewise continuous on $[0, \infty)$, then the **convolution**, $f * g$, is defined by the integral

$$f * g = \int_0^t f(t - \tau)g(\tau)d\tau.$$

2. Convolution is commutative, i.e. $f * g = g * f$
3. Convolution Theorem. If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ exist for $s \geq a > 0$ then for $s > a$

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\} = F(s)G(s),$$

or

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g.$$

4. Use the convolution integral to compute

(a) $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}$

$$(b) \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 1)^2} \right\}$$

5. Consider IVP:

$$y'' + \omega^2 y = g(t), \quad y(0) = 0, \quad y'(0) = 1.$$

(a) Express the solution of the given IVP in terms of the convolution integral.

- (b) Use the Method of Variation of Parameters to solve the given IVP and compare the result with (a).