22: Systems of FIRST Order Equations. Preliminaries. (chapter 7)

1. First-order system of DE:
$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \dots, x_n) \\ x_2' &= F_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \dots, x_n) \end{aligned} \tag{1}$$

2. A set of differentiable functions $x_1(t), x_2(t), \dots, x_n(t)$ satisfying the system (1) is called a **solution** of the system (1).

3. System of ODE using a **vector notation**:
$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F_1(t, x_1, x_2, \dots, x_n) \\ F_2(t, x_1, x_2, \dots, x_n) \\ \vdots \\ F_n(t, x_1, x_2, \dots, x_n) \end{pmatrix}$$
Then the system (1) can be written as
$$\mathbf{X}' = \mathbf{F}(t, \mathbf{X}). \tag{2}$$

4. Consider the following DE of unforced undamped vibration:
$$y'' + y = 0. \tag{3}$$
subject to the initial conditions
$$y(0) = 1, \quad y'(0) = -3.$$
   (a) Transform (3) into a system of first order ODE. Is the obtained system autonomous?
   (b) Find the solution of the system obtained in item (a) under the given initial conditions.
   (c) Discuss phase portrait.

5. More generally, any scalar DE equation of order n,
$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)})$$
can be transformed to a system of n DE of the first order by introducing derivatives up to order $n-1$ as new variables.

6. To transform the following n-th order IVP,

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}),$$

$$(t_0) = \alpha_0, \quad y'(t_0) = \alpha_1, \ldots, \quad y^{(n-1)}(t_0) = \alpha_{n-1}$$

into the system we set

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \\ &\vdots \\ x_n(t) &= y^{(n-1)}(t) \end{aligned}$$

to get

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_n' &= f(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

subject to

$$x_1(t_0) = \alpha_0, \quad x_2(t_0) = \alpha_1, \ldots, \quad x_n(t_0) = \alpha_{n-1}.$$

7. Note, if f depends on t then the system is called **non-autonomous** and the phase portrait (space) in this case is in $\mathbb{R}^{n+1}$. Otherwise (i.e. if f doesn't depend on t)the system is **autonomous** and the phase portrait (space) in this case is in $\mathbb{R}^n$.

8. Important: Not any system of n first order ODE comes from a scalar n-th order.

9. **Existence and Uniqueness Theorem** for IVP defined by a system: Consider the IVP:

$$\begin{array}{rcl} x_1' &=& F_1(t, x_1, x_2, \ldots, x_n) \\ x_2' &=& F_2(t, x_1, x_2, \ldots, x_n) \\ &\vdots& \\ x_n' &=& F_n(t, x_1, x_2, \ldots, x_n) \\ x_1(t_0) &=& x_1^0 \\ x_2(t_0) &=& x_2^0 \\ &\vdots& \\ x_n(t_0) &=& x_n^0 \end{array} \tag{4}$$

If each of the functions $F_1, F_2, \ldots, F_n$ and the partial derivatives $\frac{\partial F_1}{\partial x_k}, \frac{\partial F_2}{\partial x_k}, \ldots, \frac{\partial F_n}{\partial x_k}$ $(1 \le k \le n)$ are continuous in a region

$$R = \{\alpha < t < \beta, \alpha_1 < x_1 < \beta_1, \alpha_2 < x_1 < \beta_2, \ldots, \alpha_n < x_n < \beta_n\}$$

and the point $(t_0, x_1^0, \ldots, x_n^0)$ belongs to R, then there is an interval $(t_0 - h, t_0 + h)$ in which there exists a unique solution of the IVP (4).

Linear Systems

10. When each of the functions $F_1, F_2, \dots, F_n$ in (4) is linear in the dependent variables $x_1, \dots, x_n$, we get a system of linear equations:

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t) \end{aligned} \tag{5}$$

When $g_k(t) \equiv 0$ $(1 \le k \le n)$, the linear system (5) is said to be **homogeneous**; otherwise it is **nonhomogeneous.**

11. In the previous example (3), the system

$$x_1' = x_2$$
$$x_2' = -x_1$$

is linear homogeneous system of DE, which is also autonomous with constant coefficients:

$$p_{11} = p_{22} = 0, \quad p_{12} = 1, \quad p_{21} = -1.$$

12. **Existence and Uniqueness Theorem** for linear IVP: *If the functions $p_{11}, p_{12}, \dots, p_{nn}$ and $g_1, \dots, g_n$ are continuous on an open interval $I = \{t : \alpha < t < \beta\}$, then there exists a unique solution of the system (5) that also satisfies the initial conditions $x_1(t_0) = x_1^0, x_2(t_0) = x_2^0, \dots, x_n(t_0) = x_n^0$, where t_0 is any point of I. Moreover, the solution exists throughout the interval I.*

Matrix Form of A Linear System

13. If X, $P(t)$, and $G(t)$ denote the respective matrices

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2n}(t) \\ \vdots & & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \dots & p_{nn}(t) \end{pmatrix}, \quad G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

then the system of linear first-order DE (5) can be written as

$$X' = PX + G.$$

If the system is homogeneous, its matrix form is then

$$X' = PX.$$

14. *Example.* Express the given system in matrix form:

(a) $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned}$$

(b) $$\begin{aligned} x_1' &= x_2 - x_1 + t \\ x_2' &= -x_1 + 7x_2 - x_3 - e^t \\ x_3' &= 2x_2 - x_3 + \sin t \end{aligned}$$