

22: Systems of FIRST Order Equations. Preliminaries. (chapter 7)

1. First-order system of DE:

$$\begin{aligned}x'_1 &= F_1(t, x_1, x_2, \dots, x_n) \\x'_2 &= F_2(t, x_1, x_2, \dots, x_n) \\&\vdots \\x'_n &= F_n(t, x_1, x_2, \dots, x_n)\end{aligned}\tag{1}$$

2. A set of differentiable functions $x_1(t), x_2(t), \dots, x_n(t)$ satisfying the system (1) is called a **solution** of the system (1).

3. System of ODE using a **vector notation**:

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F_1(t, x_1, x_2, \dots, x_n) \\ F_2(t, x_1, x_2, \dots, x_n) \\ \vdots \\ F_n(t, x_1, x_2, \dots, x_n) \end{pmatrix}$$

Then the system (1) can be written as

$$\mathbf{X}' = \mathbf{F}(t, \mathbf{X}).\tag{2}$$

4. Consider the following DE of unforced undamped vibration:

$$y'' + y = 0.\tag{3}$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = -3.$$

- Transform (3) into a system of first order ODE. Is the obtained system autonomous?
- Find the solution of the system obtained in item (a) under the given initial conditions.
- Discuss phase portrait.

5. More generally, any scalar DE equation of order n ,

$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)})$$

can be transformed to a system of n DE of the first order by introducing derivatives up to order $n - 1$ as new variables.

6. To transform the following n -th order IVP,

$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)}),$$

$$(t_0) = \alpha_0, \quad y'(t_0) = \alpha_1, \dots, \quad y^{(n-1)}(t_0) = \alpha_{n-1}$$

into the system we set

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \\ &\vdots \\ x_n(t) &= y^{(n-1)}(t) \end{aligned}$$

to get

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= x_3 \\ &\vdots \\ x'_n &= f(t, x_1, x_2, \dots, x_n) \end{aligned}$$

subject to

$$x_1(t_0) = \alpha_0, \quad x_2(t_0) = \alpha_1, \dots, \quad x_n(t_0) = \alpha_{n-1}.$$

7. Note, if f depends on t then the system is called **non-autonomous** and the phase portrait (space) in this case is in \mathbb{R}^{n+1} . Otherwise (i.e. if f doesn't depend on t) the system is **autonomous** and the phase portrait (space) in this case is in \mathbb{R}^n .

8. **Important:** Not any system of n first order ODE comes from a scalar n -th order.

9. **Existence and Uniqueness Theorem** for IVP defined by a system: Consider the IVP:

$$\begin{aligned} x'_1 &= F_1(t, x_1, x_2, \dots, x_n) \\ x'_2 &= F_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ x'_n &= F_n(t, x_1, x_2, \dots, x_n) \\ x_1(t_0) &= x_1^0 \\ x_2(t_0) &= x_2^0 \\ &\vdots \\ x_n(t_0) &= x_n^0 \end{aligned} \tag{4}$$

If each of the functions F_1, F_2, \dots, F_n and the partial derivatives $\frac{\partial F_1}{\partial x_k}, \frac{\partial F_2}{\partial x_k}, \dots, \frac{\partial F_n}{\partial x_k}$ ($1 \leq k \leq n$) are continuous in a region

$$R = \{\alpha < t < \beta, \alpha_1 < x_1 < \beta_1, \alpha_2 < x_2 < \beta_2, \dots, \alpha_n < x_n < \beta_n\}$$

and the point $(t_0, x_1^0, \dots, x_n^0)$ belongs to R , then there is an interval $(t_0 - h, t_0 + h)$ in which there exists a unique solution of the IVP (4).

Linear Systems

10. When each of the functions F_1, F_2, \dots, F_n in (4) is linear in the dependent variables x_1, \dots, x_n , we get a system of linear equations:

$$\begin{aligned} x'_1 &= p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + g_1(t) \\ x'_2 &= p_{21}(t)x_1 + p_{22}(t)x_2 + \dots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x'_n &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \dots + p_{nn}(t)x_n + g_n(t) \end{aligned} \quad (5)$$

When $g_k(t) \equiv 0$ ($1 \leq k \leq n$), the linear system (5) is said to be **homogeneous**; otherwise it is **nonhomogeneous**.

11. In the previous example (3), the system

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= -x_1 \end{aligned}$$

is linear homogeneous system of DE, which is also autonomous with constant coefficients:

$$p_{11} = p_{22} = 0, \quad p_{12} = 1, \quad p_{21} = -1.$$

12. **Existence and Uniqueness Theorem** for linear IVP: *If the functions $p_{11}, p_{12}, \dots, p_{nn}$ and g_1, \dots, g_n are continuous on an open interval $I = \{t : \alpha < t < \beta\}$, then there exists a unique solution of the system (5) that also satisfies the initial conditions $x_1(t_0) = x_1^0, x_2(t_0) = x_2^0, \dots, x_n(t_0) = x_n^0$, where t_0 is any point of I . Moreover, the solution exists throughout the interval I .*

Matrix Form of A Linear System

13. If X , $P(t)$, and $G(t)$ denote the respective matrices

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2n}(t) \\ \vdots & & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \dots & p_{nn}(t) \end{pmatrix}, \quad G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

then the system of linear first-order DE (5) can be written as

$$X' = PX + G.$$

If the system is homogeneous, its matrix form is then

$$X' = PX.$$

14. *Example.* Express the given system in matrix form:

$$(a) \quad \begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned}$$

$$(b) \quad \begin{aligned} x_1' &= x_2 - x_1 + t \\ x_2' &= -x_1 + 7x_2 - x_3 - e^t \\ x_3' &= 2x_2 - x_3 + \sin t \end{aligned}$$