24: Basic Theory of Systems of First Order Linear Equations (sec. 7.4)

System of homogeneous linear equations

1. Consider a system of first order linear homogeneous DE:

$$X' = P(t)X. \tag{1}$$

Superposition Principle: *If the vector functions X_1 and X_2 are solutions of the homogeneous system (1), then the linear combination $C_1X_1 + C_2X_2$ is also a solution for any constants C_1, C_2.*

2. Given the following system

$$X' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} X$$

Show that the vector functions

$$X_1 = \begin{pmatrix} 2\cos t \\ -\cos t + \sin t \\ -2\cos t - 2\sin t \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

are solutions of the given system. Discuss a linear combination of these solutions.

3. Consider IVP

$$X' = P(t)X, \quad X(t_0) = b. \tag{2}$$

By Superposition Principle, if the vector functions

$$X_1(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \ldots, X_n(t) = \begin{pmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

are solutions of the homogeneous system (1), then the linear combination

$$X(t) = C_1X_1(t) + \ldots + C_nX_n(t)$$

is also a solution for any constants $C_1, \ldots, C_n$.

Question: How to determine the the constants $C_1, \ldots, C_n$ corresponding to the given IVP?

4. Consider the matrix, whose columns are vectors $X_1(t), \ldots, X_n(t)$:

$$\Psi(t) = \begin{pmatrix} x_{11}(t) & \ldots & x_{1n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & \ldots & x_{nn}(t) \end{pmatrix}.$$

Then $B = X(t_0) = C_1X_1(t_0) + \ldots + C_nX_n(t_0) = \Psi(t_0)\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$, equivalently,

$$\Psi(t_0)\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = b.$$

We can find a solution for any initial condition given by vector column B if and only if $det\Psi(t_0) \neq 0$.

5. Note that this determinant is called the **Wronskian** of the solutions $X_1, \ldots, X_n$ and is denoted by
$$W\left[X_1, \ldots, X_n\right](t) = \det \Psi(t).$$

6. Note that by analogy with section 3.2, $det\Psi(t_0) \neq 0$ implies $det\Psi(t) \neq 0$ for any t.

7. If $det\Psi(t) \neq 0$ then $X_1, \ldots, X_n$ is called the **fundamental set of solutions** and the general solution of the system (1) is $C_1X_1(t) + \ldots + C_nX_n(t)$.

8. Given that the vector functions $X_1 = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$ and $X_2 = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$ are solutions of the system $X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$. Find general solution of these system.

9. Question: *How to find a fundamental set of solutions?* In the next section we answer it for the case $P(t) = const$, i.e. for system of linear homogeneous equations with constant coefficients.

10. Find general solution of $X' = AX$, where $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.