25: Systems of Linear Algebraic Equations. Eigenvalues and Eigenvectors (section 7.3)

Eigenvalues and Eigenvectors

1. A number λ is called an **eigenvalue** of matrix A if there exists a **nonzero** vector v such that
$$Av = \lambda v,$$
and v is called an **eigenvector** corresponding to the eigenvalue λ.

2. Example. If A is diagonal matrix,
$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$
then the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues and the vectors
$$v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \quad v_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$
are the corresponding eigenvectors.

3. How to find eigenvalues?

   Eigenvalue are solutions of the following **characteristic equation (polynomial):**
$$\det(A - \lambda I) = 0.$$

4. Show that the characteristic equation in the case $n = 2$ can be found as
$$\lambda^2 - \operatorname{trace}(A)\lambda + \det(A) = 0.$$

5. *Remark.* For $n \times n$ matrix the characteristic equation is a polynomial equation of degree n. The eigenvectors corresponding to λ can be found by solving the corresponding system of linear equations $(A - \lambda I)v = 0$ (as we will see in the next examples).

6. *Example.* Find eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$.

7. *Example.* Given
$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

(a) Find eigenvalues of A.

(b) Find eigenvectors of A (use Gauss Elimination Method below).

Gauss Elimination Method

8. The Gauss Method is a suitable technique for solving systems of linear equations of any size. A sequence of operations (see below) of the Gauss-Jordan elimination method allows us to obtain at each step an equivalent system - that is, a system having the same solution as the original system.

   The operations of the Gauss-Jordan elimination method are

   (a) Interchange any two equations.

   (b) Replace an equation by a nonzero multiple of itself.

   (c) Replace an equation by itself plus a nonzero multiple of any other equation.

9. An **augmented matrix** that is formed by combining the coefficient matrix and the constant matrix. For example, for the system of linear equations $\begin{cases} 3x_1 & + & 12x_2 & = & 20 \\ 2x_2 & = & x_1 & +7 & \end{cases}$ the augmented matrix is $\left(\begin{array}{cc|c} 3 & 12 & 20 \\ 1 & -2 & -7 \end{array}\right)$.

10. The goal of the Gauss Elimination Method is to get the augmented matrix into **Reduced Echelon Form**. A matrix is in row echelon form if

    - All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (all zero rows, if any, belong at the bottom of the matrix).
    - The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
    - All entries in a column below a leading entry are zeroes (implied by the first two criteria).

11. To put a matrix in Reduced Form, there are three valid <u>Row Operations:</u>

    (a) Interchange any two rows ($R_i \leftrightarrow R_j$).

    (b) Replace any row by a nonzero constant multiple of itself ($R_i \leftrightarrow cR_i$).

    (c) Replace any row by the sum of that row and a constant multiple of any other row $R_i \leftrightarrow (R_i + cR_j)$.