28: Repeated Eigenvalues (sec. 7.5 (continued) and sec. 7.8)

Preliminary

1. Definition One says that n vectors $v_1, \ldots, v_n$ constitute a basis in $\mathbb{R}^n$ if any other vector v in $\mathbb{R}^n$ can be uniquely represented as a linear combination of $v_1, \ldots, v_n$, i.e. there exist constants $c_1, \ldots, c_n$ such that
$$v = c_1 v_1 + \ldots c_n v_n.$$

Repeated eigenvalues

2. Recal that A is $n \times n$ real matrix. If m is a positive integer and $(\lambda - \lambda_k)^m$ is a factor of the characteristic polynomial $\det(A - \lambda I)$ while $(\lambda - \lambda_k)^{m+1}$ is not a factor, then λ_k is said to be an **eigenvalue of multiplicity** m. Note that $m \leq n$.

3. Let A has n distinct real eigenvalues (i.e. all eigenvalues are of multiplicity one). In this case the corresponding eigenvectors $v_1, \ldots, v_n$ constitute a basis in $\mathbb{R}^n$. Recall that a fundamental set of solutions of system $X' = AX$ in this case is $\left\{e^{\lambda_1 t} v_1, \ldots, e^{\lambda_n t} v_n\right\}$.

Case 1: There is a Basis of Eigenvectors

4. If for a given matrix A with real eigenvalues $\lambda_1, \ldots, \lambda_n$ (where some of them may be of multiplicity 2 or more, i.e. may be repeated several times) there exists a basis $v_1, \ldots, v_n$ of eigenvectors then $\left\{e^{\lambda_1 t} v_1, \ldots, e^{\lambda_n t} v_n\right\}$ is a fundamental set of solutions of system $X' = AX$.

5. Basis of eigenvectors does not always exist. For example, if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then there is no basis of eigenvectors of this matrix.

6. Basis of eigenvectors always exists for a special kind of matrix know as a **symmetric** matrix: $A^T = A$, or equivalently, $a_{ij} = a_{ji}$ for all i, j.

7. Example. *Find general solution of the system*

$$\begin{aligned} x_1' &= 3x_1 + 2x_2 + 4x_3 \\ x_2' &= 2x_1 + 2x_3 \\ x_3' &= 4x_1 + 2x_2 + 3x_3 \end{aligned}$$

Case2: There is NO Basis of Eigenvectors

8. Suppose that λ is an eigenvalue of **multiplicity two** and that there is only one eigenvector v associated with this value. In other words, there is no basis of eigenvectors of A. A second solution for a fundamental set can be found in the form

$$te^{\lambda t}v + e^{\lambda t}w,$$

where w is so called generalized eigenvector satisfying the condition

$$(A - \lambda I)w = v.$$

9. Fundamental set of solutions in the case of repeated eigenvalues for $n = 2$:

$$\left\{e^{\lambda t}, te^{\lambda t}v + e^{\lambda t}w\right\}$$

10. Example. *Consider the system:*

$$\begin{aligned} x_1' &= -3x_1 + \tfrac{5}{2}x_2 \\ x_2' &= -\tfrac{5}{2}x_1 + 2x_2 \end{aligned}$$

   (a) Find general solution of the system.

(b) Find solution of the system satisfying $x_1(0) = 2, x_2(0) = 1$.