

28: Repeated Eigenvalues (sec. 7.5 (continued) and sec. 7.8)

Preliminary

1. Definition One says that n vectors v_1, \dots, v_n constitute a basis in \mathbb{R}^n if any other vector v in \mathbb{R}^n can be uniquely represented as a linear combination of v_1, \dots, v_n , i.e. there exist constants c_1, \dots, c_n such that

$$v = c_1v_1 + \dots + c_nv_n.$$

Repeated eigenvalues

2. Recall that A is $n \times n$ real matrix. If m is a positive integer and $(\lambda - \lambda_k)^m$ is a factor of the characteristic polynomial $\det(A - \lambda I)$ while $(\lambda - \lambda_k)^{m+1}$ is not a factor, then λ_k is said to be an **eigenvalue of multiplicity** m . Note that $m \leq n$.
3. Let A has n distinct real eigenvalues (i.e. all eigenvalues are of multiplicity one). In this case the corresponding eigenvectors v_1, \dots, v_n constitute a basis in \mathbb{R}^n . Recall that a fundamental set of solutions of system $X' = AX$ in this case is $\{e^{\lambda_1 t}v_1, \dots, e^{\lambda_n t}v_n\}$.

Case 1: There is a Basis of Eigenvectors

4. If for a given matrix A with real eigenvalues $\lambda_1, \dots, \lambda_n$ (where some of them may be of multiplicity 2 or more, i.e. may be repeated several times) there exists a basis v_1, \dots, v_n of eigenvectors then $\{e^{\lambda_1 t}v_1, \dots, e^{\lambda_n t}v_n\}$ is a fundamental set of solutions of system $X' = AX$.
5. Basis of eigenvectors does not always exist. For example, if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then there is no basis of eigenvectors of this matrix.

6. Basis of eigenvectors always exists for a special kind of matrix known as a **symmetric** matrix:
 $A^T = A$, or equivalently, $a_{ij} = a_{ji}$ for all i, j .

7. *Example. Find general solution of the system*

$$x'_1 = 3x_1 + 2x_2 + 4x_3$$

$$x'_2 = 2x_1 + 2x_3$$

$$x'_3 = 4x_1 + 2x_2 + 3x_3$$

Case2: There is NO Basis of Eigenvectors

8. Suppose that λ is an eigenvalue of **multiplicity two** and that there is only one eigenvector v associated with this value. In other words, there is no basis of eigenvectors of A . A second solution for a fundamental set can be found in the form

$$te^{\lambda t}v + e^{\lambda t}w,$$

where w is so called **generalized eigenvector** satisfying the condition

$$(A - \lambda I)w = v.$$

9. Fundamental set of solutions in the case of repeated eigenvalues for $n = 2$:

$$\{e^{\lambda t}, te^{\lambda t}v + e^{\lambda t}w\}$$

10. **Example.** Consider the system:

$$\begin{aligned}x'_1 &= -3x_1 + \frac{5}{2}x_2 \\x'_2 &= -\frac{5}{2}x_1 + 2x_2\end{aligned}$$

- (a) Find general solution of the system.

(b) Find solution of the system satisfying $x_1(0) = 2, x_2(0) = 1$.