## 29: The Phase Plane: Linear Systems (section 9.1)

- 1. We consider here a non-singular  $2 \times 2$  matrix A (det  $A \neq 0$ ). In this case AX = 0 implies X = 0. Points where AX = correspond to the equilibrium (constant) solutions of system X' = AX, and they are called *critical points*. It follows that X = 0 is the only critical point of the system system X' = AX.
- 2. Solution of system X' = AX are combinations of eigenvectors  $v_1, v_2$  with coefficients depending on the parameter t. This solution is also a vector functions of t. Such functions can be viewed as a parametric representation for a curve in the  $x_1x_2$ -plane. We regard to this curve as the path, or **trajectory**, traversed by a moving particle whose velocity X'(t) is specified by the differential equation. The plane  $x_1x_2$  itself is called the **phase plane**, and a representative set of trajectories is referred to as a **phase portrait**.

### Case 1. Real Distinct Eigenvalues

3. General solution  $(\lambda_1, \lambda_2)$  are eigenvalues and  $v_1, v_2$  are corresponding eigenvectors):

$$X(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2.$$
(1)

Coordinates of X(t) in the basis  $\{v_1, v_2\}$  are

$$(C_1 e^{\lambda_1 t}, C_2 e^{\lambda_2 t}) =: (\xi_1(t), \xi_2(t))$$

Eliminating the parameter t, one get

$$\xi_2 = C\xi_1^{\lambda_2/\lambda_1}.$$

#### Case 1a: Real Distinct Eigenvalues of the Same Sign

4. Example. Sketch several trajectories in the phase plane for the system

$$\begin{aligned}
x_1' &= -2x_1 + x_2 \\
x_2' &= 2x_1 - 3x_2
\end{aligned}$$
(2)

Previously we obtained<sup>1</sup>

$$\lambda_1 = -4, \quad \lambda_2 = -1, \quad v_1 = \begin{pmatrix} -1\\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1\\ 1 \end{pmatrix}.$$

<sup>&</sup>lt;sup>1</sup>see set 26(#3) of notes

 $\mathbf{2}$ 

5. Sketch several trajectories in the phase plane when both eigenvalues are positive.

## Case 1b: Real Eigenvalues of the Opposite Sign

6. Example. Sketch several trajectories in the phase plane for the system

$$\begin{cases} x'_1 = x_1 + 2x_2 \\ x'_2 = 4x_1 + 3x_2 \end{cases}$$
(3)

Previously we obtained<sup>2</sup>

$$\lambda_1 = -1, \quad \lambda_2 = 5, \quad v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

# Case 2: Complex Eigenvalues

7. General solution ( $\lambda = \alpha + i\beta$  is a complex eigenvalue and v = a + ib is a corresponding eigenvector):

$$X(t) = C_1 \operatorname{Re}(e^{\lambda t}v) + C_2 \operatorname{Im}(e^{\lambda t}v) = C_1 e^{\alpha t} (a\cos(\beta t) - b\sin(\beta t)) + C_2 e^{\alpha t} (a\sin(\beta t) + b\cos(\beta t)).$$

8. Sketch several trajectories in the phase plane in the case  $\alpha = 0$  (i.e.  $\lambda$  is pure imaginary).

9. Sketch several trajectories in the phase plane in the case  $\alpha < 0$ .

10. Sketch several trajectories in the phase plane in the case  $\alpha > 0$ 

## **Case 3: Repeated Eigenvalues**

#### Case 3a: There is Basis of Eigenvectors

11. General solution ( $\lambda$  is eigenvalue and  $v_1, v_2$  are corresponding eigenvectors):

$$X(t) = C_1 e^{\lambda t} v_1 + C_2 e^{\lambda t} v_2.$$

Coordinates of X(t) in the basis  $\{v_1, v_2\}$  are

$$(C_1 e^{\lambda t}, C_2 e^{\lambda t}) =: (\xi_1(t), \xi_2(t)).$$

Eliminating the parameter t, one get

$$\xi_2 = \frac{C_2}{C_1} \xi_1.$$

12. Sketch several trajectories in the phase plane in this case.

#### Case 3b: There is NO Basis of Eigenvectors

13. General solution ( $\lambda$  is eigenvalue of multiplicity 2, v is a corresponding eigenvector, and w is a generalized eigenvector):

$$X(t) = C_1 e^{\lambda t} v + C_2 (t e^{\lambda t} v + e^{\lambda t} w)$$

14. Example. Sketch several trajectories in the phase plane for the system

$$\begin{array}{rcl} x_1' &=& -3x_1 + \frac{5}{2}x_2 \\ x_2' &=& -\frac{5}{2}x_1 + 2x_2 \end{array}$$

Previously we obtained<sup>3</sup>

$$\lambda = -\frac{1}{2}, \quad v = \begin{pmatrix} 1\\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0\\ 2/5 \end{pmatrix}.$$

<sup>&</sup>lt;sup>3</sup>see set 28(#10) of notes

# Summary

Eigenvalues, $\lambda$	Type of Critical Point	Stability
$\lambda_1 > \lambda_2 > 0$	Proper node	Unstable
$\lambda_1 < \lambda_2 < 0$	Proper node	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Saddle point	Unstable
$\lambda_{1,2} = \alpha \pm i\beta,  \alpha > 0$	Spiral source	Unstable
$\lambda_{1,2} = \alpha \pm i\beta,  \alpha < 0$	Spiral sink	Asymptotically stable
$\lambda_{1,2} = \alpha \pm i\beta,  \alpha = 0$	Center	Stable
$\lambda_1 = \lambda_2 > 0$	Proper or Improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or Improper node	Asymptotically stable

15. Stability properties of linear systems X' = AX with  $det(A - \lambda I) = 0$  and  $det A \neq 0$ .