30: Non-homogeneous Linear Systems (section 7.9)

1. Consider a Non-homogeneous Linear system

$$X' = P(t)X + G(t), \tag{1}$$

where

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2n}(t) \\ \vdots & & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \dots & p_{nn}(t) \end{pmatrix}, \quad G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

2. By analogy with the case of a scalar equation we have here that general solution can be expressed as

$$X(t) = X_p(t) + X_H(t),$$

where X_p is a particular solution of the given system, and X_h is a solution of the corresponding homogeneous system,

$$X' = P(t)X. \tag{2}$$

3. Suppose that $\{X_1(t), \dots, X_n(t)\}$ is a fundamental set solutions of the corresponding homogeneous system (2). Consider the so called **fundamental matrix**, $\Psi(t)$, whose columns are vectors $X_1(t), \dots, X_n(t)$ [1]

$$\Psi(t) = \begin{pmatrix} x_{11}(t) & \dots & x_{1n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & \dots & x_{nn}(t) \end{pmatrix}.$$

Then

$$X_h(t) = \Psi(t)C, \tag{3}$$

where $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$.

Method of Variation of Parameters

4. We use the above method to find $X_p(t)$. Seek a particular solution of (1) variating parameters:

$$X(t) = \Psi(t)U(t),$$

where $U(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$.

[1] Recall that $\det \Psi(t) = W[X_1, \dots, X_n](t)$

5. One can show that then
$$\Psi(t)U'(t) = G(t),$$
which implies
$$X_p(t) = \int_0^t \Psi(t)\Psi^{-1}(\tau)g(\tau)\mathrm{d}\tau.$$
As a result we get general solution of (1):
$$X(t) = \Psi(t)C + \int_0^t \Psi(t)\Psi^{-1}(\tau)g(\tau)\mathrm{d}\tau.$$

6. Example Find general solution of the system:
$$\begin{aligned} x_1' &= -2x_1 + x_2 + e^{-t} \\ x_2' &= x_1 - 2x_2 - e^{-t} \end{aligned}$$
where $0 < t < \pi$.