4. Modeling with First Order ODE (section 2.3)

1. An ODE that describes some physical process is called **mathematical model.**

2. Reminder: **A model of free-fall motion retarded by air resistance** was considered in Section 1 (#13) of these notes.

3. **Mixing Problems** serve as a model for problem of discharge and filtration of pollutants in a river, injection and absorption of medication in the bloodstream, for example.

4. *Chemicals in Tank, or Mixing.* Assume that a tank contains V gal of water and Q_0 lb of some substance (salt, or chemical, for example) is dissolved in it. Suppose that the water containing $\gamma(t)$ lb/gal of substance is entering the tank with the rate r gal/min (the concentration of the chemicals in the entering water varies in time) and then well stirred mixture is draining from the tank at the same rate. Find the amount $Q(t)$ of the substance in tank at any time (Do not solve, just find IVP for $Q(t)$).

5. *Chemicals in Pond* (from the textbook) A pond contains 10 million gal of fresh water. Stream water containing an undesirable chemical flows into the pond at the rate 5 million gal/yr, and the mixture in the pond flows out through an overflow culvert at the same rate. The concentration $\gamma(t)$ of chemical in the incoming water varies periodically with time t, measured in years, according to the expression $\gamma(t) = 2 + \sin(2t)$g/gal.

   (a) Construct a mathematical model of this flow process.
   (b) Determine the amount of chemical in the pond at any time.
   (c) Plot the solution and describe in words the effect of the variation of the incoming concentration.

5. Direction Field for $y' = f(t, y)$ (section 1.1)

6. QUALITATIVE ANALYZING of a first-order ODE $\dfrac{dy}{dt} = f(t, y)$.

7. By **Uniqueness Theorem** for $\dfrac{dy}{dt} = f(t, y)$ (see Section 2.4 later) if $f(t, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous, then the integral curves do not intersects. In other words, if we start with (t_0, y_0) then the solution is predefined.

8. At each point (t, y) on an integral curve of $y' = f(t, y)$, the tangent line has the slope $f(t, y)$.

9. To sketch direction field use the following steps:

   - Choose a rectangular grid of points in the ty-plane.
   - Calculate the slopes of tangent lines to the integral curves at the gridpoints.
   - Draw a short line segment of the tangent lines through the gridpoints.

   Note: More gridpoints $\Longrightarrow$ better description of integral curves (general shape of solution).

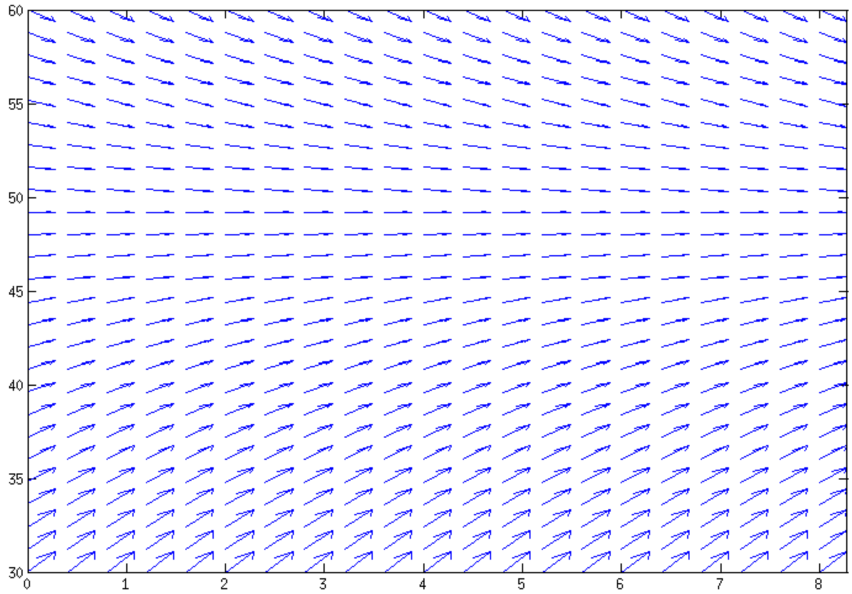
10. *FALLING HAILSTONE* Consider hailstone with mass $m = 0.03kg$ and drag coefficient $\gamma = 0.006kg/s$.

(a) Write down the ODE describing the motion of the hailstone.

(b) Qualitatively (i.e without solving the ODE) investigate the behavior of the solution from a geometrical viewpoint (Use Matlab).

Solution. To determine the qualitative behavior of obtained ODE we can proceed. We can use Matlab to get directional field:

```
[t,v]=meshgrid(0:0.4:8, 30:1.2:60);
S=9.8-0.2*v;
quiver(t,v,ones(size(S)),S), axis tight
```



11. (a) Sketch a direction field for $y' = 1 - 2y$.
    (b) Using the previous item, qualitatively describe the behavior of the solution as $t \to \pm\infty$.
    (c) Compare your answer in (b) using the analytic solution of the given equation.

6. Autonomous equations, and population dynamics (sections 1.1 and 2.5 combined)

12. Equation $y' = f(t, y)$ is called **autonomous system** if f does not depend on time t. In this case we have $y' = f(y)$.

13. Given the differential equation:
$$y' = y - y^2 \tag{1}$$
    (a) Find all equilibrium points.
    (b) Sketch a direction field.
    (c) Based on the sketch of the direction field from the item (b) answer the following questions:

i. Let $y(t)$ be the solution of equation (1) satisfying the initial condition $y(0) = \frac{1}{2}$. Find the limit of $y(t)$ when $t \to +\infty$ and the limit of $y(t)$ when $t \to -\infty$ (for this you do not need to find $y(t)$ explicitly).

ii. Find all y_0 such that the solution of the equation (1) with the initial condition $y(0) = y_0$ has the same limit at $+\infty$ as the solution from the item (c)i.

iii. Let $y(t)$ be the solution of equation (1) with $y(0) = -1$. Decide wether $y(t)$ is monotonically decreasing or increasing and find to what value it approaches when t increases (the value might be infinite).

(d) Find the solution of the equation (1) with $y(0) = -1$ explicitly. Determine the interval in which this solution is defined.

14. **Classification of equilibrium points**: Suppose $f(y_0) = 0$, i.e y_0 is an equilibrium points.

    Unstable: $f(y)$ changes sign from "$-$" to "$+$".

    Stable: $f(y)$ changes sign from "$+$" to "$-$".

    Semistable: f does not change sign at y_0.

15. For autonomous systems the slope on horizontal lines $y = y_0$ is the same and the qualitative analysis can be made on the so called *phase line portrait.*

16. Directions to draw phase line portrait:

    - Find all equilibrium points (i.e. roots of $f(y) = 0$), draw a horizontal line and mark those points on it.
    - Check the sign of $f(y)$ in each of the intervals determined by the equilibrium points. Over those intervals where $f(y) > 0$ draw arrows pointing to the right, and on those where $f(y) < 0$ draw arrows pointing to the left (indicating in which direction are solutions flowing).

17. Carry out a phase line analysis for the equation (1).

18. *Exponential GROWTH* Let $y = y(t)$ be the population of the given species at time t. Hypothesis: $\frac{\mathrm{d}y}{\mathrm{d}t} = ry$, where r is called the **rate of growth**.

19. *LOGISTIC GROWTH:* the growth rate depends on the population (replace r by a function $h(y)$): $\frac{\mathrm{d}y}{\mathrm{d}t} = h(y)y$

    Verhulst's model: $h(y) = a - by$ $(a, b > 0)$