7. Differences between Linear and Nonlinear Equations (section 2.4)

1. **Existence and Uniqueness of Solutions**

   THEOREM 1. *Let the functions f and $\partial f/\partial y$ be continuous in some rectangle*

   $$R = \{(t, y)|\alpha < t < \beta, \quad \gamma < y < \delta\}$$

   containing the point (t_0, y_0). Then in some interval $t_0 - h < t < t_0 + h$ contained in $I = \{t|\ \alpha < t < \beta\}$, there is a **unique** *solution $y = y(t)$ of the initial value problem*

   $$y' = f(t, y), \quad y(t_0) = y_0.$$

2. By this theorem we can guarantee the existence of solution only for values of t which are sufficiently closed to t_0, but not for all t.

3. Geometric consequence of the theorem is that two integral curves never intersect each other.

4. The condition "$\partial f/\partial y$ be continuous in some rectangle..." is important for uniqueness. Illustration: Apply the existence and uniqueness theorem to the following IVP:

   $$y' = y^{2/3}, \quad y(0) = 0.$$

5. **Existence and Uniqueness of Solutions of <u>Linear</u> ODE**

   THEOREM 2. *If the functions $p(t)$ and $g(t)$ are continuous on the interval $I = \{t|\ \alpha < t < \beta\}$, then for any $t = t_0$ on I, there is a* **unique** *solution $y = y(t)$ of the initial value problem*

   $$y' + p(t)y = g(t), \quad y(t_0) = y_0. \tag{1}$$

6. Note that the conditions of the Theorem 1 hold automatically for linear ODE.

7. Determine (without solving the problem) an interval in which the solution of the given IVP is certain to exist:

   $$(t - 3)y' + (\ln |t|)y = 2t$$

   (a) $y(1) = 2$
   (b) $y(5) = 7$

8. Consider

   $$ty' + 2y = 4t^2 \tag{2}$$

   (a) Determine (without solving the problem) an interval in which the solution (2) satisfying $y(t_0) = y_0$ with $t_0 > 0$ is certain to exist.

   (b) The solution of IVP from item (a) found by the method of integrating factor is given:

   $$y(t) = t^2 + \frac{C}{t^2}, \qquad C = t_0^2(y_0 - t_0^2).$$

   Using that information discuss the domain of the solution and compare your conclusion with the answer of (a).