8. Exact Equations and Integrating Factors (section 2.6)

1. Method for solving the first order ODE

$$M(x,y) + N(x,y)y' = 0 \quad (\text{or} \quad M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0) \tag{1}$$

   for the special case in which (1) represents the **exact differential** of a function $z = \Phi(x,y)$.

2. The equation (1) is an **exact ODE** if there exists a function $\Phi(x,y)$ having continuous partial derivatives such that

$$\Phi_x(x,y) = M(x,y), \qquad \Phi_y(x,y) = N(x,y).$$

   and The general solution of the equation is $\Phi(x,y) = C$ (geometrically, the integral curve $y = y(x)$ lies on a level curve of the function $z = \Phi(x,y)$.)

3. *TEST for Exactness:* If M and N have continuous partial derivatives then the ODE (1) is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2}$$

4. Question: *Is every separable equation exact?*

5. *Determine whether the following ODE are exact:*

(a) $3x^2 - 2xy + 2 + (6y^2 - x^2 + 3)y' = 0$

(b) $(3x^2y + 2xy + y^3)\mathrm{d}x + (x^2 + y^2)\mathrm{d}y = 0$

6. `Test for exactness and Conservative Vector Field:`

The test for exactness is the same as the test for determining whether a vector field $\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$ is conservative. Namely, it is the same as the test for determining whether $\mathbf{F}(x, y) =$ the gradient of a potential function.

`CONCLUSION`: *A general solution to an exact differential equation can be found by the method used to find a potential function for a conservative vector field.*

7. Solve $3x^2 - 2xy + 2 + (6y^2 - x^2 + 3)y' = 0$

8. **A nonexact ODE made exact:** If ODE is not exact, it may be possible to make it exact by multiplying by an appropriate integrating factor.

- If $\dfrac{M_y - N_x}{N}$ is a function of x alone, then an integrating factor for (1) is
$$\mu(x) = e^{\int \frac{M_y - N_x}{N} \mathrm{d}x}.$$
- If $\dfrac{N_x - M_y}{M}$ is a function of y alone, then an integrating factor for (1) is
$$\mu(y) = e^{\int \frac{N_x - M_y}{M} \mathrm{d}y}.$$

9. Solve $(3x^2y + 2xy + y^3)\mathrm{d}x + (x^2 + y^2)\mathrm{d}y = 0$