9. Solutions of linear homogeneous equations of second order. The Wronskian (section 3.2).

1. Second order ODE:

$$y'' = f(t, y, y')$$

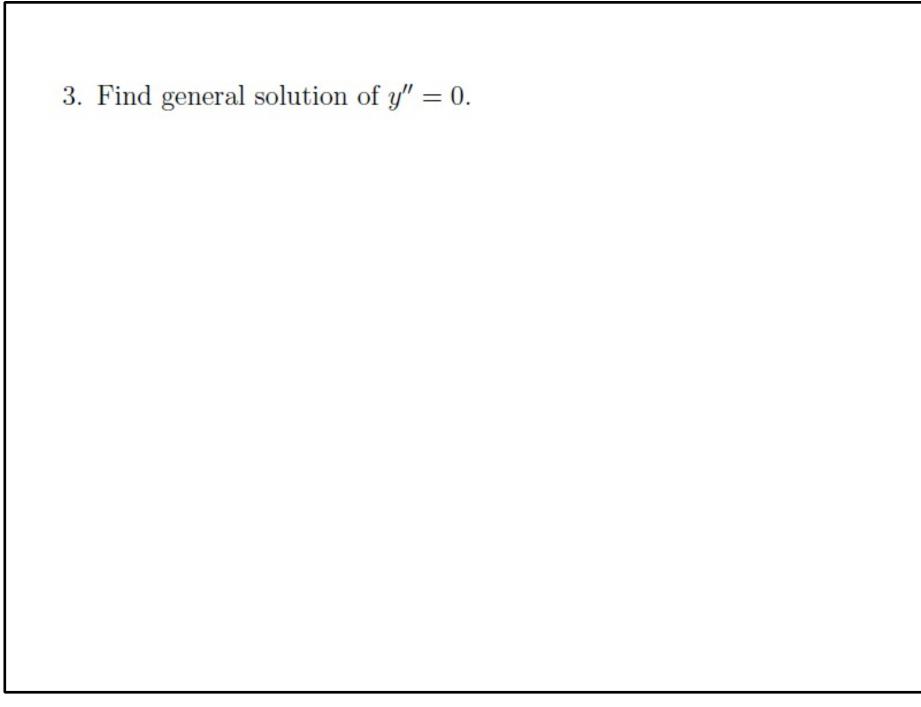
where f is a function whose domain is in \mathbb{R}^3 . A solution of such an equation is a function y = y(t), twice differentiable in some interval I, s.t. for all t in I

$$y''(t) = f(t, y(t), y'(t)).$$

2. By Newton's Second Law F = ma. We know that acceleration a(t) is the second derivative of position x(t), which yields

$$x'' = \frac{1}{m}F(x, x').$$

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4.	Reduction	to	the	first	order	ODE	by	a	substitution:
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- Particular case y'' = f(t, y') use the substitution u = y'.
- General case y'' = f(t, y, y') use the substitution $x_1 = y$, $x_2 = y'_1$.

5. By analogy with the existence and uniqueness theorem for a single first order ODE we have

THEOREM 1. If f, $\frac{\partial f}{\partial x_1}$ and, $\frac{\partial f}{\partial x_2}$ are continuous in a region

$$R = \{ \alpha < t < \beta, \quad \alpha_1 < x_1 < \beta_1, \quad \alpha_2 < x_2 < \beta_2 \}$$

then there exists a unique solution through a point (t_0, y_0, v_0) in R (equivalently, there is an interval $t_0 - h < t < t_0 + h$ in which there exists a unique solution of the IVP $y'' = f(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = v_0.$)

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6. Second order linear ODE:

$$y'' + p(t)y' + q(t)y = g(t)$$

or

$$y'' = -p(t)y' - q(t)y + g(t)$$

where p, q, and g are continuous on an interval $I = (\alpha, \beta)$

7. By analogy with the existence and uniqueness theorem for a linear first order ODE we have

THEOREM 2. If the functions p(t), q(t) and g(t) are continuous on the interval then for any $t = t_0$ on I, there is a unique solution y = y(t) of the IVP

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = v_0.$$
(1)

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Linear HOMOGENEOUS ODE of second order

8. Question: Can the function $y = \sin(t^2)$ be a solution on the interval (-1,1) of a second order linear homogeneous equation with constant coefficients?

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9. Consider a linear homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0 (2)$$

with coefficients p and q continuous in an interval I.

- 10. Superposition Principle
 - Sum $y_1(t) + y_2(t)$ of any two solutions $y_1(t)$ and $y_2(t)$ of (2) is itself a solution.

• A scalar multiple Cy(t) of any solution y(t) of (2) is itself a solution.

COROLLARY 3. Any linear combination $C_1y_1(t) + C_2y_2(t)$ of any two solutions $y_1(t)$ and $y_2(t)$ of (2) is itself a solution.

11. Why Superposition Principle is important? Once two solutions of a linear homogeneous equation are known, a whole class of solutions is generated by linear combinations of these two.

12. WRONSKIAN of the functions $y_1(t)$ and $y_2(t)$:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix}$$

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13. Suppose that $y_1(t)$ and $y_2(t)$ are two differentiable solutions of (2) in the interval I such that $W(y_1, y_2)(t) \neq 0$ somewhere in I, then every solution is a linear combination of $y_1(t)$ and $y_2(t)$.

In other words, the family of solutions $y(t) = C_1y_1(t) + C_2y_2(t)$ with arbitrary coefficients C_1 and C_2 includes every solution of (2) if and only if there is a points t_0 where $W(y_1, y_2)$ is not zero. In this case the pair $(y_1(t), y_2(t))$ is called the **fundamental** set of solutions of (2).

REMARK 4. Wronskian $W(y_1, y_2)(t)$ (of any two solutions $y_1(t)$ and $y_2(t)$ of (2))either is zero for all t or else is never zero.

14. Confirm that $\sin x$ and $\cos x$ are solutions of y'' + y = 0. Then solve the IVP

$$y'' + y = 0$$
, $y(1) = 0$, $y'(1) = -5$

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Appendix: Facts from Algebra

1. • FACT 1: Cramer's Rule for solving the system of equations

$$a_1x + b_1y = c_1$$
$$a_1x + b_2y = c_2$$

The rule says is that if the determinant of the coefficient matrix is not zero, i.e.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

then the system has a unique solution (x, y) given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

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- FACT 2: If determinant of the coefficient matrix is zero then either there is no solution, or there are infinitely many solutions.
- FACT 3. The homogeneous system of linear equations

$$a_1x + b_1y = 0$$

$$a_1x + b_2y = 0$$

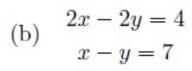
always has the "trivial" solution (x, y) = (0, 0). By Cramer's rule this is the only solution if the determinant of the coefficient matrix is not zero.

• FACT 4: If determinant of the coefficient matrix of homogeneous system of linear equations is zero then there are infinitely many nontrivial solutions $(x, y) \neq (0, 0)$.

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2.	Use Facts 1-4 to determine if each the following systems of linear equations has one solution, no solution
	infinitely many solutions. Then find the solution/s (if any).

(a)
$$2x + 3y = 5$$
$$x - y = 4$$



(c)
$$2x - 2y = 0$$
$$3x + 3y = 0$$

(d)
$$2x - 2y = 0$$
$$3x - 3y = 0$$