10. Linear homogeneous equations of second order with constant coefficients the case of distinct roots of characteristic equation (sec. 3.1)

1. PROBLEM: Find general solutions of linear homogeneous equation

$$ay'' + by' + cy = 0 \tag{1}$$

with constant real coefficients a, b, and c.

2. Recall that

   - By Superposition Principle: *Any* `linear combination` $C_1y_1(t) + C_2y_2(t)$ *of any two solutions* $y_1(t)$ *and* $y_2(t)$ *of (1) is itself a solution.*
   - The family of solutions $y(t) = C_1y_1(t) + C_2y_2(t)$ with arbitrary coefficients C_1 and C_2 includes every solution of (1) if and only if there is a points t_0 where $W(y_1, y_2)$ is not zero. In this case the pair $(y_1(t), y_2(t))$ is called the **fundamental set** of solutions of (1).

3. To find a fundamental set for the equation (1) note that the nature of the equation suggests that it may have solutions of the form

$$y = e^{rt}.$$

Plug in and get so called **characteristic equation** of (1):

$$ar^2 + br + c = 0. \tag{2}$$

Note that the characteristic equation can be determined from its differential equation simply by replacing $y^{(k)}$ with with r^k.

4. *Solve* $y'' - 16y = 0$.

5. `Fact from Algebra:` The quadratic equation $ar^2 + br + c = 0$ has roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

which fall into one of 3 cases:

   - two distinct real roots $r_1 \neq r_2$ (in this case $D = b^2 - 4ac > 0$) [section 3.1]
   - two complex conjugate roots $r_1 = \overline{r_2}$ (in this case $D = b^2 - 4ac < 0$) [section 3.3]
   - two equal real roots $r_1 = r_2$ (in this case $D = b^2 - 4ac = 0$) [section 3.4]

Case 1: Two distinct real roots ($D = b^2 - 4ac > 0$)

6. Two distinct real roots r_1 and r_2 of the characteristic equation give us two solutions

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t} \quad (r_1 \neq r_2).$$

Is this a fundamental set of solutions?

7. *Consider*

$$3y'' - y' - 2y = 0.$$

   (a) *Find general solution.*

   (b) *Find solution satisfying the following initial conditions:* $y(0) = \alpha, \quad y'(0) = 1$, *where* α *is a real parameter.*

8. *Consider*

$$y'' + (a + 1)y' + (a - 2)(1 - 2a)y = 0$$

   where a *is a real parameter.*

   (a) *Determine the values of the parameter* a*, if any, for which all solutions tend to zero as* $t \to \infty$.

   (b) *Determine the value of the parameter* a*, if any, for which all (nonzero) solutions become unbounded as* $t \to \infty$.