

11: Complex roots of the characteristic equation (section 3.3.)

1. **Case 2:** two complex conjugate roots $r_1 = \bar{r}_2$ (in this case $D = b^2 - 4ac < 0$)
2. Recall that the **characteristic equation** of a linear homogeneous equation with constant real coefficients

$$ay'' + by' + cy = 0 \quad (1)$$

is

$$ar^2 + br + c = 0. \quad (2)$$

$$D = b^2 - 4ac < 0 \Rightarrow r_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = -\frac{b}{2a} \pm i \frac{\sqrt{|D|}}{2a} =: \lambda + i\mu$$

Then two particular solutions which form a fundamental set are

$$y_1 = e^{(\lambda+i\mu)t}, \quad y_2 = e^{(\lambda-i\mu)t}.$$

3. Note that formally there is no difference between this case and Case 1 (two distinct real roots). However in practice we prefer to work with real functions instead of complex exponentials. By Superposition Principle the following linear combinations are solutions as well

$$\frac{1}{2}(y_1 + y_2) = e^{\lambda t} \cos(\mu t), \quad \frac{1}{2i}(y_1 - y_2) = e^{\lambda t} \sin(\mu t)$$

Note that these solutions are real functions.

4. FACT: $\frac{d}{dt}(e^{rt}) = re^{rt}$ for any complex r .
5. Show that $\{e^{\lambda t} \cos(\mu t), e^{\lambda t} \sin(\mu t)\}$ is a fundamental set of solutions, i.e. that general solution of (1) has a form

$$y(t) = C_1 e^{\lambda t} \cos(\mu t) + C_2 e^{\lambda t} \sin(\mu t) \quad (3)$$

6. Solve the following two differential equations which are important in applied mathematics:

$$y'' + \omega^2 y = 0 \quad \text{and} \quad y'' - \omega^2 y = 0,$$

where ω is a real positive constant.

7. **Alternative form of solution (3):**

$$y(t) = e^{\lambda t} R \cos(\mu t - \delta), \quad (4)$$

where

$$R = \sqrt{C_1^2 + C_2^2}, \quad \cos \delta = \frac{C_1}{\sqrt{C_1^2 + C_2^2}} = \frac{C_1}{R}, \quad \sin \delta = \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \frac{C_2}{R}.$$

Note that $\tan \delta = C_2/C_1$.

8. Application: Mechanical unforced vibration: a mass hanging from a spring (more details in Section 3.7).

- $\lambda = 0$ corresponds to **undamped** free vibration (simple harmonic motion)
- $\lambda < 0$ corresponds to **damped** free vibration
- R is called the **amplitude** of the motion
- δ is called the **phase**, or phase angle, and measures the displacement of the wave from its normal position corresponding to $\delta = 0$.
- $T = \frac{2\pi}{\mu}$ is the **period** of the motion.

9. Consider

$$y'' + 2y' + 3y = 0. \quad (5)$$

- (a) Find general solution.
- (b) Find solution of (5) subject to the initial conditions

$$y(0) = 2, \quad y'(0) = 1.$$

- (c) Sketch the graph of the solution of IVP from (b) and describe its behavior as t increases.