11: Complex roots of the characteristic equation (section 3.3.)

1. **Case 2:** two complex conjugate roots $r_1 = \overline{r_2}$ (in this case $D = b^2 - 4ac < 0$)

2. Recall that the **characteristic equation** of a linear homogeneous equation with constant real coefficients
$$ay'' + by' + cy = 0 \tag{1}$$
is
$$ar^2 + br + c = 0. \tag{2}$$
$$D = b^2 - 4ac < 0 \Rightarrow r_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = -\frac{b}{2a} \pm i\frac{\sqrt{|D|}}{2a} =: \lambda + i\mu$$
Then two particular solutions which form a fundamental set are
$$y_1 = e^{(\lambda+i\mu)t}, \qquad y_2 = e^{(\lambda-i\mu)t}.$$

3. Note that formally there is no difference between this case and Case 1 (two distinct real roots). However in practice we prefer to work with real functions instaed of complex exponentials. By Superposition Principle the following linear combinations are solutions as well
$$\frac{1}{2}(y_1 + y_2) = e^{\lambda t}\cos(\mu t), \quad \frac{1}{2i}(y_1 - y_2) = e^{\lambda t}\sin(\mu t)$$
Note that these solutions are real functions.

4. FACT: $\frac{\mathrm{d}}{\mathrm{d}t}(e^{rt}) = re^{rt}$ for any complex r.

5. Show that $\left\{e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)\right\}$ is a fundamental set of solutions, i.e. that general solution of (1) has a form
$$y(t) = C_1 e^{\lambda t}\cos(\mu t) + C_2 e^{\lambda t}\sin(\mu t) \tag{3}$$

6. Solve the following two differential equations which are important in applied mathematics:

$$y'' + \omega^2 y = 0 \quad \text{and} \quad y'' - \omega^2 y = 0,$$

where ω is a real positive constant.

7. **Alternative form of solution (3)**:

$$y(t) = e^{\lambda t} R \cos(\mu t - \delta), \tag{4}$$

where

$$R = \sqrt{C_1^2 + C_2^2}, \quad \cos\delta = \frac{C_1}{\sqrt{C_1^2 + C_2^2}} = \frac{C_1}{R}, \quad \sin\delta = \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \frac{C_2}{R}.$$

Note that $\tan\delta = C_2/C_1$.

8. Application: Mechanical unforced vibration: a mass hanging from a spring (more details in Section 3.7).

- $\lambda = 0$ corresponds to **undamped** free vibration (simple harmonic motion)
- $\lambda < 0$ corresponds to **damped** free vibration
- R is called the **amplitude** of the motion

- δ is called the **phase**, or phase angle, and measures the displacement of the wave from its normal position corresponding to $\delta = 0$.
- $T = \dfrac{2\pi}{\mu}$ is the **period** of the motion.

9. Consider

$$y'' + 2y' + 3y = 0. \tag{5}$$

(a) Find general solution.

(b) Find solution of (5) subject to the initial conditions

$$y(0) = 2, \quad y'(0) = 1.$$

(c) Sketch the graph of the solution of IVP from (b) and describe its behavior as t increases.