12. The case of equal (or repeated) roots (section 3.4)

1. **Case 3:** two equal/repeated real roots $r_1 = r_2 (= r)$ (in this case $D = b^2 - 4ac = 0$)

2. Recall that the **characteristic equation** of a linear homogeneous equation with constant real coefficients
$$ay'' + by' + cy = 0 \quad (1)$$
is
$$ar^2 + br + c = 0. \quad (2)$$
$$D = b^2 - 4ac = 0 \Rightarrow r = -\frac{b}{2a}$$
So, we found one particular solution $y_1(t) = e^{rt}$.

3. How to find a second particular solution y_2 such that the set $\{y_1, y_2\}$ be fundamental, i.e. $W(y_1, y_2) \neq 0$?

   Use "factorization" (in Leibnitz notation):
$$ay'' + by' + cy = a\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = \left(a\frac{\mathrm{d}^2}{\mathrm{d}t^2} + b\frac{\mathrm{d}}{\mathrm{d}t} + c\right)[y] = a\left(\frac{\mathrm{d}}{\mathrm{d}t} - r\right)\left[\left(\frac{\mathrm{d}}{\mathrm{d}t} - r\right)[y]\right] = 0$$

4. $\{y_1, y_2\} = \{e^{rt}, te^{rt}\}$ is fundamental set. Thus the general solution of (3) is
$$y(t) = C_1 e^{rt} + C_2 t e^{rt}.$$

5. Consider $y'' - 6y' + 9y = 0$.
   (a) Find general solution.
   (b) Find solution subject to the initial conditions $y(0) = 2, y'(0) = \alpha$.
   (c) For each α what is the behavior of the solutions as $t \to +\infty$?

SUMMARY:

Solution of linear homogeneous equation of second order with constant coefficients

$$ay'' + by' + cy = 0 \tag{3}$$

Sign of $D = b^2 - 4ac$	Roots of characteristic polynomial $ar^2 + br + c = 0$	General solution
$D > 0$	two distinct real roots $r_1 \neq r_2$	$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
$D < 0$	two complex conjugate roots $r_1 = \overline{r_2}$: $r_{1,2} = \lambda \pm i\mu$	$y(t) = C_1 e^{\lambda t} \cos(\mu t) + C_2 e^{\lambda t} \sin(\mu t)$
$D = 0$	two equal(repeated) real roots $r_1 = r_2 = r$	$y(t) = C_1 e^{rt} + C_2 t e^{rt}$