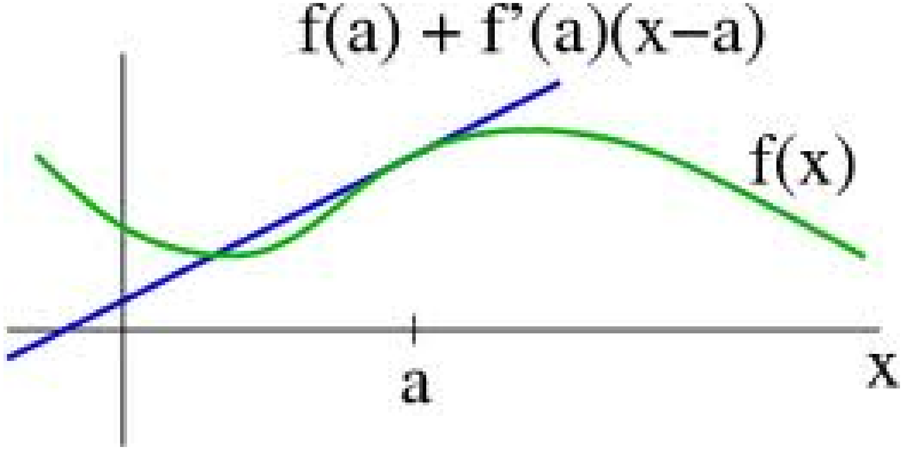
DIFFERENTIALS: LINEAR AND QUADRATIC APPROXIMATION

By Kenny Abitbol, Sam Mote, and Evan Kirkland

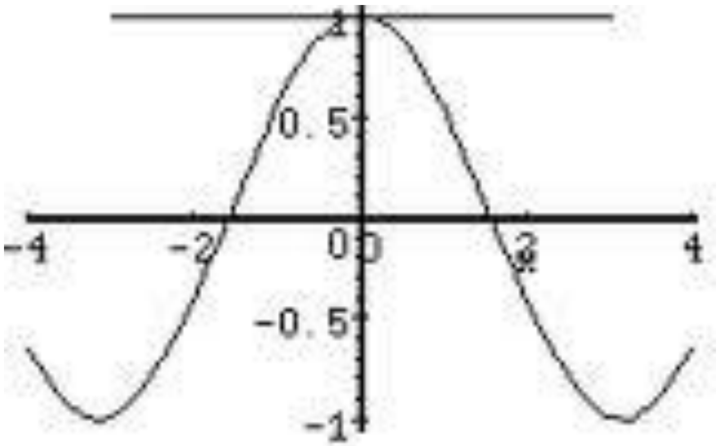


Introduction of Linear Approximation

$$L(x) = f(a) + f'(a)(x - a)$$

- a : point whose tangent line as an approximation of the function
- x : point whose value is being approximated
- If $f(x)$ is concave up after $x = a$, the approximation will be an underestimate, and if it is concave down, it will be an overestimate

Linear approximation of $f(x) = \cos x$ at $a = 0$.



Definition of Linear Approximation

"The equation of the tangent line to the curve $y = f(x)$ at $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a),$$

so…the tangent line at $P(a, f(a))$ [is] an approximation to the curve $y = f(x)$ when x is near a.

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a."

- Stewart, "Calculus: Early Vectors"

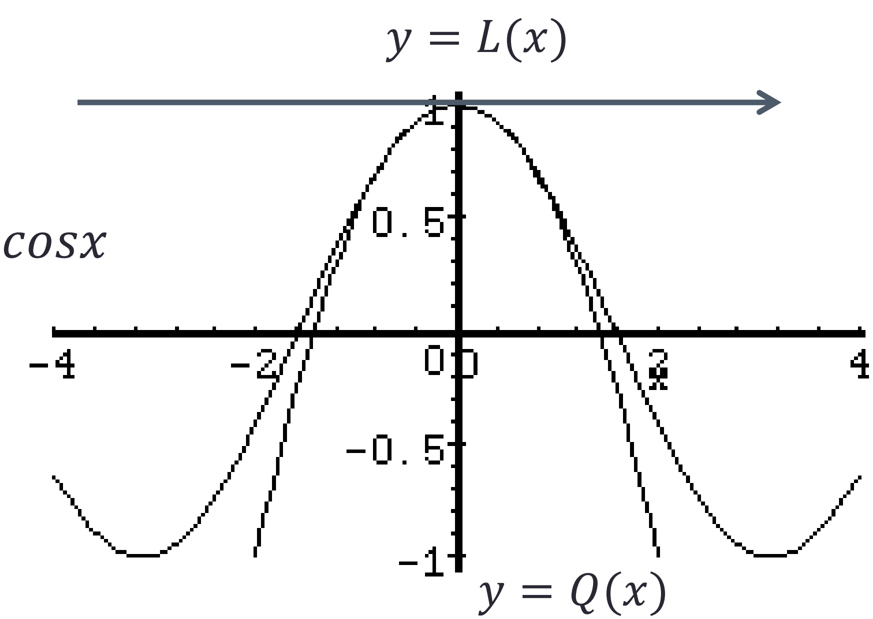
Introduction of Quadratic Approximation

$$Q(x) = f(a) + f'(a)(x-a) + (\tfrac{1}{2})f''(a)(x-a)^2$$

or

$$Q(x) = L(x) + (\tfrac{1}{2})f''(a)(x-a)^2$$

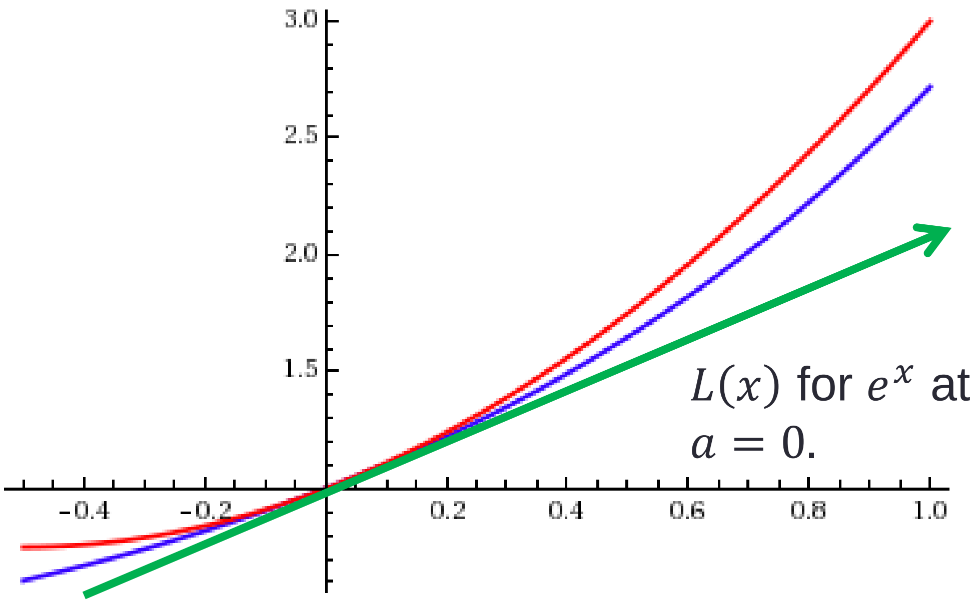
Quadratic approximation of $f(x) = cosx$ at $a = 0$.



Definition of Quadratic Approximation

The quadratic approximation also uses the point $x = a$ to approximate nearby values, but uses a parabola instead of just a tangent line to do so.

This gives a closer approximation because the parabola stays closer to the actual function.



History: Taylor's Theorem

Linear and Quadratic approximations are based off of Taylor's theorem of polynomials. The theorem is named after 18th century mathematician Brook Taylor who designed a general formula for approximating the values of functions after a small change of the x-value. The formula was first published in 1712. His theorem is:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k \quad ,$$

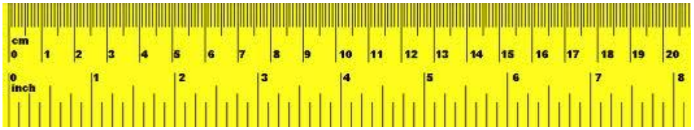
where **a** is a reference point, **x** is the nearby value being approximated, and **k** is the maximum amount of times the original function can be derived.

History: Differentials

- Began in 1680’s
  - Bernouli brothers
  - Gottfried Wilhelm Leibniz
- Applied to geometry and mechanics

Producing Along Standards

A company requires that the bowling balls that it creates have a volume in the range of $4170 - 4200cm^3$. If the radii of the balls are produced to be 10 cm and have an error of .01 cm, will the company be able to produce along these standards?



Error Calculation with Linear Approximation

$$V = \frac{4}{3}\pi r^3$$

$$L(x) = f(a) + f'(a)(x - a)$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$f(x) = f(a) + f'(a)(x - a)$$

$$\frac{dV}{dr} = 4\pi 10^2$$

$$f(x) - f(a) = f'(a)(x - a)$$

$$\Delta y = f'(a)\Delta x$$

$$\Delta V = \frac{dV}{dr}\Delta r$$

Error Calculation with Linear Approximation

Now using the formula $\Delta V = \frac{dV}{dr}\Delta r$

$$\frac{dV}{dr} = 400\pi$$

$$\Delta r = 0.01$$

$$\Delta V = 400\pi(.01) = 12.57cm^3$$

$$V(10) = \frac{4}{3}\pi 10^3 = 4188.79cm^3$$

Analysis

We found that the volume of the perfectly created ball would be $4188.79cm^3$ with a deviation of $\pm 12.57cm^3$. This creates a range of $4176.22 - 4201.26cm^3$, which does not fit with the desired range of $4170 - 4200cm^3$. The company will not be able to produce along these standards.

Surface Area Errors

In the design of a waterpark fountain like the one shown below, the cost of paint is important. With the paint costing $\$0.25/inches^2$, what would the relative error in surface area be if the radius of the circle has an error of $\pm .75\ inches$?What would be the variation in price?

Take the inner radius to be $\frac{R}{4}$,

the radius of the circle to be $37\ inches$,

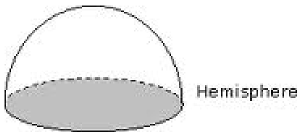
and the height to be $2\mathrm{R}$.

Modifying the Linear Approximation Formula

$$L(x) = f(a) + f'(a)(x-a) \rightarrow \Delta y = \frac{dy}{dx}\Delta x$$

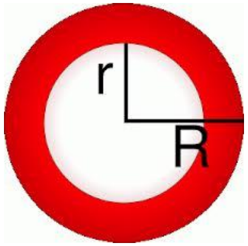
Adding up the surface areas:

Sphere: $2\pi R^2$



Bottom of sphere: $\pi R^2 - \pi(\frac{R}{4})^2$

Cylinder: $\pi(\frac{R}{4})^2 + 2\pi\frac{R}{4}2R$ (top of cylinder doesn't matter)



Surface Area $= 4\pi R^2$

Proposed Area $= 4\pi(37)^2$

$\underline{= 17{,}203.36\ inches^2}$

Error in Surface Area

Surface Area Error : $\Delta R = .75\ inches$

Now take the derivative of the surface area formula, with respect to the radius of the hemisphere.($A(R) = 4\pi R^2$)

$$\frac{dA}{dR} = 8\pi R,$$

Error in Surface Area

$$\Delta A_R = \frac{dA}{dR}\Delta R \rightarrow 8\pi(37)(.75) = 697.43\ inches^2$$

$$error_{relative} = \frac{\Delta A}{A} = \frac{697.43}{17203.36} = \underline{0.0405}$$

$$\Delta price = (697.43)(0.25) = \underline{\$174.36}$$

Analysis

The small error in the radius of the sphere can cause a substantial change in cost.

The cost of painting the structure: $\text{Cost} = Cost_0 \pm \Delta Cost$

$Cost_0 = (17203.36)(.25) = \$4300.84$

$\Delta Cost = \$174.36$

$\text{Cost} = \$(4300.84 \pm 174.36)$

Percent Error of R: $\frac{.75}{37} \times 100 = 2.03\%$

Percent Error of A: $\frac{697.43}{17203.36} \times 100 = 4.05\%$

Oscillation with Differentials

The water level in an artificial pond, which has both a constant drainage, due to absorption and evaporation, and a periodically active source, over a given 24-hour period is given by the approximation

$$h(x) = 5\sin\left(\frac{x}{2}\right) + 25,$$

where h is given in ft. This equation was the result of taking measurement tests every 30 minutes. That leaves the depth at many other times indefinite, but assumable. Given that $x = 0$ is midnight, $x = 1$ is 1:00 A.M., and so on, what would the depth have been at 11:03 A.M.?

Oscillation with Differentials

The first thing to do is identify given values.

$$h(x) = 5\sin\left(\frac{x}{2}\right) + 25, \qquad a = 11 \ , \qquad and$$

$$x = 11 + \left(\frac{3}{60}\right) \quad or \quad 11.05 \ .$$

And we know that the linear and quadratic approximation formulas are

$$L(x) = h(a) + h'(a)(x - a) \quad and$$

$$Q(x) = L(x) + \frac{h"}{2}(x - a)^2$$

Oscillation with Differentials

Now we plug in values and solve where we can. So if $h(x) = 5\sin\left(\frac{x}{2}\right) + 25,\ then\ h'(x) = 2.5\cos\left(\frac{x}{2}\right).$ Now :

$L(11.05) = h(11) + h'(11)(11.05 - 11)$ ➔

$L(11.05) = 21.472 + (1.772)(.05)$ ➔

$L(11.05) = 21.5606$

So according to the linear approximation, at 11:03, the water's depth was 21.5606 ft.

Oscillation with Differentials

Now to find the depth of the water according to a quadratic approximation, we need $h''(x)$. So if

$$h'(x) = 2.5\cos\left(\frac{x}{2}\right),$$

then

$$h''(x) = -1.25\sin\left(\frac{x}{2}\right).$$

Oscillation with Differentials

Now:

$Q(11.05) = L(11.05) + \frac{h''(11)}{2}(11.05 - 11)^2$ ➔

$Q(11.05) = 21.5606 + \left(\frac{.882}{2}\right)(.05)^2$ ➔

$Q(11.05) = 21.5617$

So now according to the quadratic approximation, the depth of the water was 21.5617 ft.

Oscillation with Differentials: Analysis

When $x = 11.05$ is plugged into the original equation, the depth is 21.5620 ft. To calculate error for the Linear approximation it is

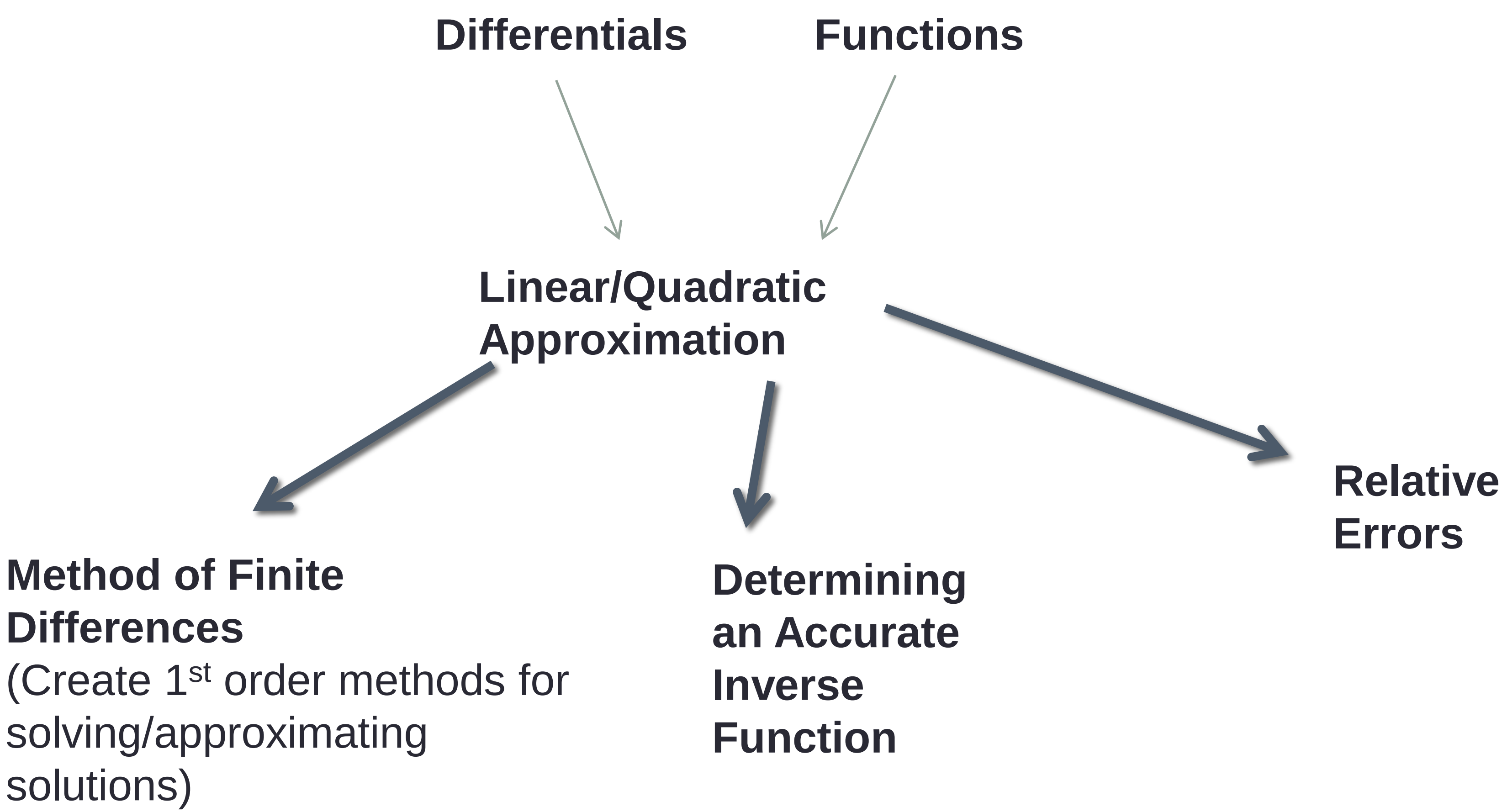
$$\frac{|h(11.05) - L(11.05)|}{h(11.05)},$$

which is an error of 0.0065%. For the Quadratic approximation error it is

$$\frac{|h(11.05) - Q(11.05)|}{h(11.05)},$$

which is an error of $.0014\%$. So while both approximations have very low error, the Quadratic approximation is even more accurate.

Visual Map



Differentials
Functions
Linear/Quadratic Approximation
Method of Finite Differences
(Create 1st order methods for solving/approximating solutions)
Determining an Accurate Inverse Function
Relative Errors

Common Mistakes

- Omission of $\frac{1}{2}$ in equation for quadratic approximation
- $(x - a)$ is used instead of $(x - a)^2$ in the quadratic approximation
- Chain rule forgotten when taking derivatives
- Plug $f'(x)$ instead of $f'(a)$ into linear or quadratic approximation

References

"Linear Approximation." Wikipedia.org. 6 November 2012. Web. 3 June 2012. <http://en.wikipedia.org/wiki/Linear_approximation>.

Stewart, James. "Calculus: Early Vectors." 1st ed. Pacific Grove: Brooks/Cole, 1999. Print.

"Taylor's Theorem." Wikipedia.org. 6 November 2012. Web. 5 November 2012. <http://en.wikipedia.org/wiki/Taylor's_theorem>.