Inverse Laplace transform of rational functions via Partial Fraction Decomposition over complex numbers

In the previous file we discussed the partial fraction decomposition of rational functions $\dfrac{P(s)}{Q(s)}$ with $\deg P(s) < \deg Q(s)$. The method consist of factoring the denominator $Q(s)$ as much as possible. We worked with real coefficients only and therefore in this factorization there are two types of factors:

1. linear (may be repeated) factor corresponding to a real root of $Q(s)$;
2. quadratic (may be repeated) factors corresponding to to a pair of complex conjugate roots of $Q(s)$.

Another idea is *to allow complex coefficients in the partial fraction decomposition*. Then *we have only one type of factors: linear (maybe repeated) factor corresponding either to a real or to a complex root.*

Then we can proceed as in cases 1 and 2 of the previous file (the cases 3 and 4 can be removed in this method). Each factor in the decomposition of $Q(s)$ gives a contribution of certain type to the partial fraction decomposition of $\dfrac{P(s)}{Q(s)}$. Below we list these contributions depending on the type of the factor and identify the inverse Laplace transform of these contributions (**in the case of (non-real) complex roots we just need to use the Euler formula to return from complex valued functions to real valued functions**):

Case 1 A non-repeated linear factor $(s-a)$ of $Q(s)$ (corresponding to the root a of $Q(s)$ of multiplicity 1) gives a contribution of the form $\dfrac{A}{s-a}$. Then $\mathcal{L}^{-1}\left\{\dfrac{A}{s-a}\right\} = Ae^{at}$;

Case 2 A repeated linear factor $(s-a)^m$ of $Q(s)$ (corresponding to the root a of $Q(s)$ of multiplicity m) gives a contribution which is a sum of terms of the form $\dfrac{A_i}{(s-a)^i}$, $1 \le i \le m$. Then

$$\mathcal{L}^{-1}\left\{\frac{A_i}{(s-a)^i}\right\} = \frac{A_i}{(i-1)!}t^{i-1}e^{at}$$

;

Example (will be solved later also using convolution) Find the inverse Laplace transform of $F(s) = \dfrac{1}{(s^2+1)^2}$.

Solution: $Q(s) = (s^2+1)^2 = (s-i)^2(s+i)^2 \Rightarrow Q(s)$ has two complex roots i and $-i$ both of multiplicity 2. Therefore the partial fraction decomposition of $F(s)$ over complex numbers has the form:

$$\frac{1}{(s^2+1)^2} = \frac{A}{s-i} + \frac{B}{(s-i)^2} + \frac{C}{s+i} + \frac{D}{(s+i)^2} \Rightarrow$$

$$1 = A(s-i)(s+i)^2+B(s+i)^2+C(s+i)(s-i)^2+D(s-i)^2 = A(s^2+1)(s-i)+B(s+i)^2+C(s^2+1)(s+i)+D(s-i)^2. \tag{1}$$

Determine A, B, C, and D:

1. Plug in $s = i \Rightarrow 1 = B(2i)^2 = -4B \Rightarrow B = -\frac{1}{4}$;
2. Plug in $s == i \Rightarrow 1 = D(-2i)^2 = -4B \Rightarrow D = -\frac{1}{4}$
3. Equate coefficients near s^3 in (1): $0 = A + C$;
4. Equate coefficient near s^2 in (1) and use items 1 and 2 above: $0 = iA - iC - \frac{1}{4} - \frac{1}{4} = i(A-C) - \frac{1}{2}$;
5. From items 3 and 4 we get the following system of equations for A and C:

$$\begin{cases} A + C = 0 \\ A - C = \frac{1}{2i} \end{cases} \Rightarrow$$

   By elimination, $A = \frac{1}{4i}$, $C = -\frac{1}{4i}$.

Therefore,

$$\frac{1}{(s^2+1)^2} = \frac{1}{4i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) - \frac{1}{4}\left(\frac{1}{(s-i)^2} + \frac{1}{(s+i)^2}\right).$$

Recall that $\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$, $\mathcal{L}^{-1}\{\frac{1}{(s-a)^2}\} = te^{at}$ for any complex a. Therefore

$$\mathcal{L}\{\frac{1}{(s^2+1)^2}\} = \frac{1}{4i}(e^{it} - e^{-it}) - \frac{1}{4}t(e^{it} + e^{-it}) \overset{\text{Euler's formulas}}{=} \frac{1}{2}\sin t - \frac{1}{2}t\cos t.$$